Number Theory

Lim Jeck

2 Apr 2017

1 Factorisation

Factorise the following:

1. $a^2 - b^2 = (a-b)(a+b)$
2. $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
3. $a^3 + b^3 = (a+b)(a^2-ab+b^2)$
4. $a^2 + ab + bc + ca = (a+b)(a+c)$

1.1 Problems

1. Find all positive integers n such that $n^2 + 12$ is a square.

   Proof. Let $n^2 + 12 = a^2$, then $(a-n)(a+n) = 12$. Note that both $a-n$ and $a+n$ have the same parity (both odd or both even), so they must both be even. The only solution is $a-n = 2, a+n = 6$ which gives $n = 2$ as the only solution. □

2. Find all positive integers n such that $n^2 + 24n + 35$ is a square.

   Proof. Let $n^2 + 24n + 35 = a^2$, then by completing the square, $(n+12)^2 - 109 = a^2$ so we have $(n+12-a)(n+12+a) = 109$ which is prime. So the only solution is $n+12-a = 1, n+12+a = 109$ which gives $n = 43$ as the only solution. □

3. Find all positive integers m, n such that $mn + m + n = 11$.

   Proof. Factorise into $(m+1)(n+1) - 1 = 11$ so $(m+1)(n+1) = 12$. Split 12 into all the possible factors $1, 2, 3, 4, 6, 12$, we get all the positive integer solutions $(m,n) = (1,5), (2,3), (3,2), (5,1)$. □

4. Find all positive integers m, n such that $2mn + 2m - 3n = 15$.

   Proof. Factorise into $(2m-3)(n+1) = 12$. Note that $2m-3$ is odd, so the only solutions are $2m-3 = 1, n+1 = 12$ and $2m-3 = 3, n+1 = 4$. This gives the only solutions $(m,n) = (2,11), (3,3)$. □

5. Find all integers m, n such that $2mn + m + n = 7$.

   Proof. To find the factorisation, we make the mn coefficient 1 by dividing everything by 2 to get $mn + \frac{1}{2}m + \frac{1}{2}n = \frac{7}{2}$. Now it is easy to factorise as $(m+\frac{1}{2})(n+\frac{1}{2}) = \frac{15}{4}$. Now multiply everything by 4 to make everything an integer: $(2m+1)(2n+1) = 15$. Remember that m, n are only integers, so they can be negative. So we have $(2m+1, 2n+1) = (1,15), (3,5), (-1,-15), (-3,-5)$ including symmetry (swapping m, n), so the only solutions are $(m,n) = (0,7), (1,2), (-1,-8), (-2,-3)$ including symmetry. □

6. (Junior 2008) Determine all primes p such that $5^p + 4p^4$ is a perfect square.

   Proof. See SMO Junior 2008 Round 2 Q5. □

2 Modulo Arithmetic

When you want to prove an equation has no solution in integers, sometimes it is better to work with the REMAINDER when the equation is divided by some n. This is called taking mod n.

We say that $a \equiv b \pmod{n}$ if $a - b$ is divisible by n. E.g.

- $2017 \equiv 7 \pmod{10}$
- $12 \equiv 5 \equiv -2 \equiv -9 \pmod{7}$
- $10 \equiv -5 \pmod{3}$

The usual rules of addition, subtraction and multiplication still holds in mod n! E.g.

- $2017 \times 2017 \equiv 2 \times 2 \equiv 4 \pmod{5}$

2.1 Problems

1. What are all the possible remainders when a square number x^2 is divided by 4?

   Proof. We just need to look at the remainders of $0^2, 1^2, 2^2, 3^2$, which are $0, 1, 0, 1$ respectively. So squares can only be 0 or 1 $\pmod{4}$. □

2. Prove that $a^2 + b^2 = 10003$ has no solutions in integers.

   Proof. Taking mod 4, we have $a^2 + b^2 \equiv 3 \pmod{4}$. But a^2, b^2 can only be 0 or 1, impossible to get $a^2 + b^2 \equiv 3 \pmod{4}$. □

3. Prove that $a^4 + b^4 = 10024$ has no solutions in integers.

   Proof. Take mod 5, we see that a^4 can only be 0 or 1 $\pmod{5}$ (just consider $0^2, 1^2, 2^2, 3^2, 4^2$). But we need $a^4 + b^4 \equiv 4 \pmod{5}$, which is impossible. □

4. Prove that $a^2 + b^2 + c^2 = 20028$ has no solutions in integers.

   Proof. Take mod 8, we see that a^2 can only be $0, 1, 4 \pmod{8}$. We need $a^2+b^2+c^2 \equiv 4 \pmod{8}$. The only possible way is to have all a, b, c to be even. Let $a = 2x, b = 2y, c = 2z$, then we have $x^2 + y^2 + z^2 = 5007$. Again take mod 8, we get $x^2 + y^2 + z^2 \equiv 7 \pmod{8}$, which is impossible. □

3 More problems

1. (Junior 2006) Find all integers x, y that satisfy the equation $x + y = x^2 - xy + y^2$.

   Proof. Write it as a quadratic in x: $x^2 - (y + 1)x + y^2 - y = 0$. Now solve for x:

   $$x = \frac{y + 1 \pm \sqrt{(y + 1)^2 - 4(y^2 - y)}}{2}.$$

   For x to be real, we need the thing inside the square root to be non-negative, so $(y + 1)^2 - 4(y^2 - y) \geq 0 \implies -3(y - 1)^2 \geq -4$. So the only possible y are $y = 0, 1, 2$. Subbing that into our equation for x, we get all the solutions $(x, y) = (0, 0), (1, 0), (0, 1), (2, 1), (1, 2), (2, 2)$. □

2. (Junior 2007) Let n be a positive integer and d be the greatest common divisor of $n^2 + 1$ and $(n + 1)^2 + 1$. Find all the possible values of d.

   Proof. We have

   $$d | n^2 + 1 \qquad (1)$$
   $$d | (n + 1)^2 + 1 \qquad (2)$$

   Take their difference $(2) - (1)$ and multiply by $(2n - 1)$:

   $$d | 2n + 1 \qquad (3)$$
   $$d | 4n^2 - 1 \qquad (4)$$

   Take $4 \times (1) - (4)$ to get $d|5$, so $d = 1, 5$ only. Both are possible values: consider $n = 1, 2$. □