

Number Theory

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1 Factorisation

Factorise the following:

1. $a^2 - b^2$
2. $a^3 - b^3$
3. $a^3 + b^3$
4. $a^2 + ab + bc + ca$

1.1 Problems

1. Find all positive integers n such that $n^2 + 12$ is a square.
2. Find all positive integers n such that $n^2 + 24n + 35$ is a square.
3. Find all positive integers m, n such that $mn + m + n = 11$.
4. Find all positive integers m, n such that $2mn + 2m - 3n = 15$.
5. Find all integers m, n such that $2mn + m + n = 7$.
6. (Junior 2008) Determine all primes p such that $5^p + 4p^4$ is a perfect square.

2 Modulo Arithmetic

When you want to prove an equation has no solution in integers, sometimes it is better to work with the REMAINDER when the equation is divided by some n . This is called taking mod n .

We say that $a \equiv b \pmod{n}$ if $a - b$ is divisible by n . E.g.

- $2017 \equiv 7 \pmod{10}$
- $12 \equiv 5 \equiv -2 \equiv -9 \pmod{7}$
- $10 \equiv -5 \pmod{3}$

The usual rules of addition, subtraction and multiplication still holds in mod n ! E.g.

- $2017 \times 2017 \equiv 2 \times 2 \equiv 4 \pmod{5}$

2.1 Problems

1. What are all the possible remainders when a square number x^2 is divided by 4?
2. Prove that $a^2 + b^2 = 10003$ has no solutions in integers.
3. Prove that $a^4 + b^4 = 10024$ has no solutions in integers.
4. Prove that $a^2 + b^2 + c^2 = 10028$ has no solutions in integers.

3 More problems

1. (Junior 2006) Find all integers x, y that satisfy the equation $x + y = x^2 - xy + y^2$.
2. (Junior 2007) Let n be a positive integer and d be the greatest common divisor of $n^2 + 1$ and $(n + 1)^2 + 1$. Find all the possible values of d .