Geometry 2

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16 Apr 2017

Now with circles!

1 Circles

"It is the set of all points (in a plane) that are at a given distance from a given point, the centre."

Consider a circle with center O and 3 points A, B, C on the circle. Then $OA = OB = OC$ (obviously).

Theorem 1 (so called "Angle at the Center Theorem"). *If O lies inside the triangle ABC, then $\angle AOB = 2\angle ACB$ (not so obvious). In fact the same result holds even if O does not lie inside ABC (you just have to be careful on where to measure the angle).*

Here is the most important fact about circles:

Theorem 2 (so called "Angles Subtended by Same Arc Theorem"). *If $ABCD$ is a quadrilateral with vertices on a circle, then $\angle ABD = \angle ACD$. In fact the converse is also true: If $ABCD$ is a quadrilateral with $\angle ABD = \angle ACD$, then there is a circle passing through A, B, C, D. We call such a quadrilateral* cyclic.

Every triangle has a unique circle passing through its 3 vertices. This circle is called the *circumcircle*, and the center of the circle is the *circumcenter* of the triangle. The circumcenter can be constructed by taking the intersection of the perpendicular bisectors of the 3 sides of the triangle (which are concurrent).

1.1 Problems

1. If $ABCD$ is a cyclic quadrilateral, prove that $\angle ABC + \angle CDA = 180°$.

   Proof. By angles subtended by the same arc, $\angle ABD = \angle ACD$ and $\angle DBC = \angle DAC$, so $\angle ABC = \angle ABD + \angle DBC = \angle ACD + \angle DAC = 180° - \angle CDA$. □

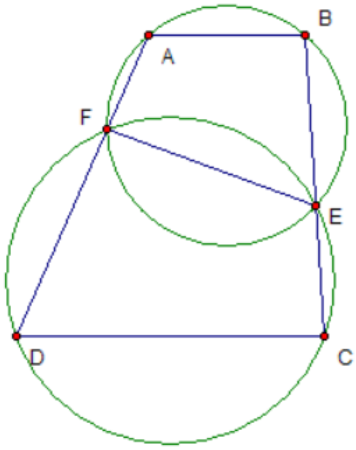
2. Let C be a point on the circle with diameter AB. Prove that $\angle ACB = 90°$.

   Proof. Note that the center of the circle O lies on AB, so by Angle at the Center Theorem, we have $angle ACB = \frac{1}{2}\angle AOB = \frac{1}{2} \times 180° = 90°$. □

3. Let ABC be a triangle and the angle bisector of $\angle A$ intersects its circumcircle at D. Prove that $DB = DC$.

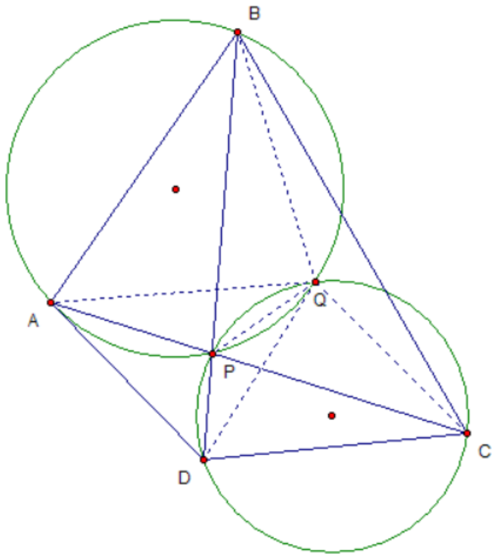
   Proof. By angles subtended by the same arc, $\angle BCD = \angle BAD = \angle DAC = \angle DBC$. Hence $\triangle DBC$ is isosceles, so $DB = DC$. □

4. Let $ABCD$ be a quadrilateral and E, F are points on sides BC, DA respectively such that $ABEF$ and $CDFE$ are cyclic. Prove that $AB \parallel CD$.



Proof. Using problem 1, we have $\angle FAB = 180 - \angle BEF = \angle FEC = 180 - \angle CDF$, so $AB \parallel DC$. □

5. Let $ABCD$ be a convex quadrilateral whose diagonals intersect at P. Suppose the circumcircles of $\triangle ABP$ and $\triangle CDP$ intersect at $Q \neq P$. Prove that $\triangle ABQ \sim \triangle CDQ$.



Proof. We shall show $\angle QBA = \angle QDC$ and $\angle BQA = \angle DQC$. Note that $\angle QBA = \angle QPC = \angle QDC$ and $\angle BQA = \angle BPA = \angle DPC = \angle DQC$. So $\triangle ABQ \sim \triangle CDQ$. □

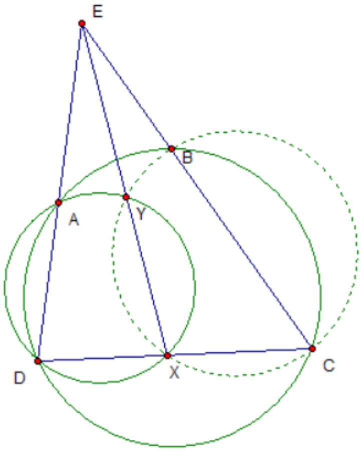
2 More on circles

Theorem 3 (Power of a Point)**.** *Let A, B, C, D be points on a circle. If AB intersects DC at E, then*

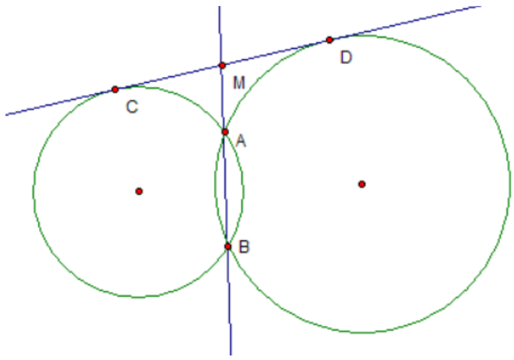
$$EA \cdot EB = EC \cdot ED.$$

What if E lies inside the circle? What if A, B are the same point?

2.1 Problems



(a) Problem 2.1



(b) Problem 2.2

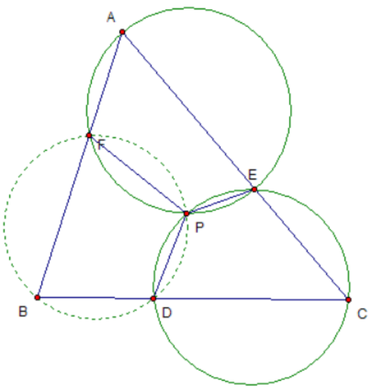
1. Let $ABCD$ be a cyclic quadrilateral and DA intersects CB at E. Let X be a point on CD and suppose the circumcircle of $\triangle ADX$ intersects EX again at Y. Prove that $BCXY$ is cyclic.

   Proof. By power of a point, $EB \cdot EC = EA \cdot ED = EY \cdot EX$, hence $BCXY$ is cyclic by (converse of) power of a point. □

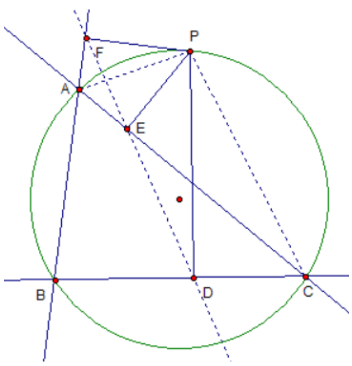
2. Two circles Γ_1, Γ_2 intersect each other at A, B. Suppose a line is tangent to both Γ_1, Γ_2 at C, D. Prove that AB bisects CD.

   Proof. Let AB intersect CD at M. Then by power of a point, $MC^2 = MA \cdot MB = MD^2$, hence M is the midpoint of CD. □

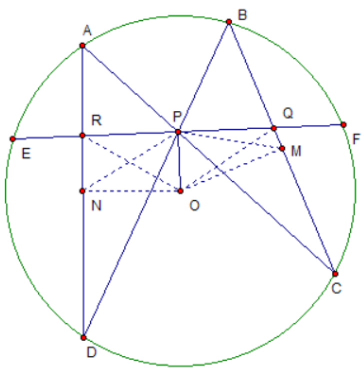
3 More Problems



(a) Problem 3.1



(b) Problem 3.2



(c) Problem 3.2

1. Let ABC be a triangle and D, E, F are points on the sides BC, CA, AB. Prove that the circumcircles of triangles AEF, BFD, CDE pass through a common point.

   Proof. Let the circumcircles of $\triangle AEF, \triangle CDE$ intersect at $P \neq E$. Then $\angle AFP = \angle CEP = \angle BDP$, so $BDPF$ is cyclic. Hence the circumcircles of triangles AEF, BFD, CDE pass through P. □

2. Let ABC be a triangle and P a point on its circumcircle. Let D, E, F be the feet of perpendiculars from P onto BC, CA, AB. Prove that D, E, F are collinear.

   The line that they lie on is called the *Simson line*.

   Proof. We shall show that $\angle FEA = \angle DEC$. Note that $\angle AFP + \angle PEA = 90 + 90 = 180$, we have $AFPE$ is cyclic. Similarly, $\angle PEC = \angle PDC$ so $PEDC$ is cyclic. Now we have

$$\begin{aligned} \angle FEA &= \angle FPA \\ &= 90 - \angle FAP \\ &= 90 - \angle BCP \\ &= \angle DPC \\ &= \angle DEC \end{aligned}$$

   and we are done. □

3. Let A, B, C, D be points occurring in that order on circle ω and let P be the point of intersection of AC and BD. Let EF be a chord of ω passing through P such that P is the midpoint of EF, Q be the point of intersection of BC and EF, and R be the point of intersection of DA and EF. Prove that $PQ = PR$.

   This is known as the *Butterfly Theorem*.

   Proof. Let M, N be the midpoints of CB, AD. Let O be the center of the circle ω. Then $OP \perp EF, OM \perp BC, ON \perp AD$. Thus $ONRP$ and $OMQP$ are cyclic. Note that $\triangle APD \sim \triangle BPC$, and since M, N are the midpoints of their same respective sides, we also have $\triangle ANP \sim \triangle AMP$. So $\angle ANP = \angle BMP$. Now we have $\angle POR = \angle PNA = \angle BMP = \angle QOP$, so $\triangle POR \equiv \triangle POQ$ and finally we have $PQ = PR$. □