Geometry 2

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Now with circles!

1 Circles

"It is the set of all points (in a plane) that are at a given distance from a given point, the centre."

Consider a circle with center O and 3 points A, B, C on the circle. Then $OA = OB = OC$ (obviously).

Theorem 1 (so called "Angle at the Center Theorem")**.** *If O lies inside the triangle ABC, then $\angle AOB = 2\angle ACB$ (not so obvious). In fact the same result holds even if O does not lie inside ABC (you just have to be careful on where to measure the angle).*

Here is the most important fact about circles:

Theorem 2 (so called "Angles Subtended by Same Arc Theorem")**.** *If $ABCD$ is a quadrilateral with vertices on a circle, then $\angle ABD = \angle ACD$. In fact the converse is also true: If $ABCD$ is a quadrilateral with $\angle ABD = \angle ACD$, then there is a circle passing through A, B, C, D. We call such a quadrilateral* cyclic.

Every triangle has a unique circle passing through its 3 vertices. This circle is called the *circumcircle*, and the center of the circle is the *circumcenter* of the triangle. The circumcenter can be constructed by taking the intersection of the perpendicular bisectors of the 3 sides of the triangle (which are concurrent).

1.1 Problems

1. If $ABCD$ is a cyclic quadrilateral, prove that $\angle ABC + \angle CDA = 180°$.
2. Let C be a point on the circle with diameter AB. Prove that $\angle ACB = 90°$.
3. Let ABC be a triangle and the angle bisector of $\angle A$ intersects its circumcircle at D. Prove that $DB = DC$.
4. Let $ABCD$ be a quadrilateral and E, F are points on sides BC, DA respectively such that $ABEF$ and $CDFE$ are cyclic. Prove that $AB \parallel CD$.
5. Let $ABCD$ be a convex quadrilateral whose diagonals intersect at P. Suppose the circumcircles of $\triangle ABP$ and $\triangle CDP$ intersect at $Q \neq P$. Prove that $\triangle ABQ \sim \triangle CDQ$.

2 More on circles

Theorem 3 (Power of a Point)**.** *Let A, B, C, D be points on a circle. If AB intersects DC at E, then*

$$EA \cdot EB = EC \cdot ED.$$

What if E lies inside the circle? What if A, B are the same point?

2.1 Problems

1. Let $ABCD$ be a cyclic quadrilateral and DA intersects CB at E. Let X be a point on CD and suppose the circumcircle of $\triangle ADX$ intersects EX again at Y. Prove that $BCXY$ is cyclic.
2. Two circles Γ_1, Γ_2 intersect each other at A, B. Suppose a line is tangent to both Γ_1, Γ_2 at C, D. Prove that AB bisects CD.

3 More Problems

1. Let ABC be a triangle and D, E, F are points on the sides BC, CA, AB. Prove that the circumcircles of triangles AEF, BFD, CDE pass through a common point.
2. Let ABC be a triangle and P a point on its circumcircle. Let D, E, F be the feet of perpendiculars from P onto BC, CA, AB. Prove that D, E, F are collinear.

   The line that they lie on is called the *Simson line*.
3. Let A, B, C, D be points occurring in that order on circle ω and let P be the point of intersection of AC and BD. Let EF be a chord of ω passing through P, Q be the point of intersection of BC and EF, and R be the point of intersection of DA and EF. Prove that $PQ = PR$.

   This is known as the *Butterfly Theorem*.