Geometry

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26 Mar 2017

1 Congruent Triangles

Two triangles are congruent if they have the same shape and size. We write $\triangle ABC \cong \triangle DEF$ when both triangles are congruent.

$\triangle ABC \cong \triangle DEF$ if and only if their corresponding sides and angles are equal, i.e. $AB = DE, BC = EF, CA = FD$ and $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$.

Tests for congruency:

- (SSS) 3 equal pairs of corresponding sides.
- (SAS) 2 equal pairs of corresponding sides and equal pair of corresponding angles in between.
- (ASA) 1 equal pair of corresponding sides and 2 equal pair of corresponding angles.
- (RHS) Both right-angled with equal hypotenuse and an equal pair of corresponding side.

2 Similar Triangles

Two triangles are similar if they have the same shape. We write $\triangle ABC \sim \triangle DEF$ when both triangles are similar.

$\triangle ABC \sim \triangle DEF$ if and only if their corresponding sides are proportional and angles are equal, i.e. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ and $\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$.

Tests for similarity:

- 2 equal corresponding angles.
- 2 corresponding sides proportional and equal corresponding angle in between.
- 3 corresponding sides proportional.

3 Examples

Example 1.
If D, E are points on sides AB, AC of $\triangle ABC$, then $DE \parallel BC \iff \frac{AD}{DB} = \frac{AE}{EC}$.

Proof.

$$\begin{aligned} DE \parallel BC &\iff \angle ADE = \angle ABC \text{ and } \angle AED = \angle ACB \\ &\iff \triangle ADE \sim \triangle ABC \\ &\iff \frac{AD}{AB} = \frac{AE}{AC} \\ &\iff \frac{AD}{DB} = \frac{AE}{EC} \end{aligned}$$

□

Example 2. (Angle Bisector Theorem)

If D is a point on BC such that AD bisects $\angle A$, then

$$\frac{AB}{BD} = \frac{AC}{CD}$$

Proof. WLOG assume $AB \leq AC$. Extend AD to E such that $CD = CE$, then $\angle AEC = \angle EDC = \angle ADB$. Since $\angle EAC = \angle DAB$, we have $\triangle AEC \sim \triangle ADB \implies \frac{AB}{BD} = \frac{AC}{CE} = \frac{AC}{CD}$. □

4 Problems

1. (Junior 01) A trapezium $ABCD$ has parallel sides AB and CD. The diagonals AC and BD intersect at E. If the areas of $\triangle CDE$ and $\triangle CDB$ are 1 and 4 respectively, find the area of the trapezium.
2. ABC is an acute triangle and AD is its altitude. Given that $DB = 4444, DA = 6666, DC = 9999$, prove that $\angle BAC = 90°$.
3. (Junior 07) Equilateral triangles ABE and BCF are erected externally on the sides AB and BC of a parallelogram $ABCD$. Prove that $\triangle DEF$ is equilateral.
4. (Junior 13) In the triangle ABC, points D, E, F are on the sides BC, CA and AB respectively such that FE is parallel to BC and DF is parallel to CA. Let P be the intersection of BE and DF, and Q the intersection of FE and AD. Prove that PQ is parallel to AB.
5. (Junior 16) In the triangle ABC, $\angle A = 90°$, the bisector of $\angle B$ meets the altitude AD at the point E, and the bisector of $\angle CAD$ meets the side CD at F. The line through F perpendicular to BC intersects AC at G. Prove that B, E, G are collinear.