Combinatorics

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There's no theory behind combinatorics, you just have to think out of the box.

1 Problems

1. In a party with some people, some pairs of them shook hands. Everybody loves the party, so after the party, each person donated \$1 for each other person he shook hands with. Prove that the total amount of dollars donated is even.

   Proof. For every pair of handshakes, the 2 people involved donated \$1 each. So the total amount of dollars donated is twice total number of pairs of handshakes, which is even.

   A good way to think of the problem is to think of it as a *graph*. That is, treat each person as a *vertex* (like a point on the plane), and draw an *edge* (like a line) between any 2 vertices if the 2 people (who are vertices) shook hands with each other. Then the amount of dollars donated by each person is the number of edges connecting to that vertex, which we call the *degree* of the vertex. This whole structure of vertices and edges is called a *graph*, and many problems can be simplified by representing it as a graph. Now the question is that the sum of all the degrees of all the vertices is even. With the same argument, the sum of all the degrees is twice the number of edges, which is even. □

2. There are 4 islands and every pair of island is linked with a bridge. Bob wants to travel from island to island in such a way that he crosses each bridge exactly once. He starts from one of the islands. Can he accomplish that? (He can travel to an island more than once. Once he decides to cross a bridge, he must cross all the way to the other side; no U-turning allowed)

   Proof. We draw a graph where each vertex represents an island and we draw an edge between any 2 vertices if there is a bridge between the 2 islands. So we get 4 vertices and 6 edges, and the degree of each vertex is 3. Now except for the island that Bob starts or ends, for every other island Bob must visit it an equal number of times as he leave it. But each time he visits or leaves an island, he needs an edge from that vertex. So the degree of that vertex must be even. But the degrees of all our vertices are odd, and Bob can only start or end at 2 different islands. So he cannot cross each bridge exactly once. □

3. (Junior 2011) Let $S_1, S_2, \dots, S_{2011}$ be nonempty sets of consecutive integers such that any 2 of them have a common element. Prove that there is an integer that belongs to every $S_i, i = 1, \dots, 2011$. (For example, $\{2, 3, 4, 5\}$ is a set of consecutive integers while $\{2, 3, 5\}$ is not.)

   Proof. Suppose each set S_i is of the form $[a_i, b_i]$, that is, it contains the integers from a_i to b_i inclusive, where $b_i > a_i$. Now WLOG assume that b_1 is the smallest among all the b_i's. We claim that b_1 is contained in every S_i. That is the same as saying $a_i \leq b_1 \leq b_i$ for all $i = 1, 2, \dots, 2011$. The latter inequality is true by our choice of b_1. Suppose $a_k > b_1$ for some k. Then the sets $S_1 = [a_1, b_1]$ lies completely to the left of $S_k = [a_k, b_k]$, so they do not intersect, a contradiction. Therefore $a_k \leq b_1$ for all k and we are done. □

4. 25 boys and 25 girls sit around a table. Prove that it is always possible to find a person both of whose neighbours are girls.

   Proof. Label the chairs around the table $1, 2, \dots, 50$. Assume that everybody's 2 neighbours are not both girls. That means the 2 sitting at chair k and $k + 2$ cannot be both girls for $k = 1, 2, \dots, 50$, with chairs 51 and 52 the same as chairs 1 and 2. Consider the odd numbered chairs $1, 3, 5, \dots, 49$. If we take those chairs (and the people on them) and put them around a new table of 25, in that new table every 2 people next to each other cannot be both girls. Suppose WLOG there is a girl at chair 1. Then among chairs 5/7, there is at most 1 girl. Among chairs 9/11, at most 1 girl. Among chairs 13/15... Among chairs

45/47, at most 1 girl. No girls at chairs 3/49. So there are at most 12 girls among the odd-numbered chairs. Similarly at most 12 girls among the even-numbered chairs, so total there can only be at most 24 girls, a contradiction. □

5. (Junior 2015) Let $\overline{30x070y03}$ be a 9-digit integer. Find all possible value of the pair (x, y), so that n is a multiple of 37.

   Proof. Note that $\overline{30x070y03} \equiv 300070003 + 1000000x + 100y \pmod{37}$, and we can simplify it further by finding the remainders of 300070003, 1000000 and 100 when divided by 37 by just long division. But a faster way is to note that 37 divides 999, so $1000 \equiv 1 \pmod{37}$. So $300070003 \equiv 373 \equiv 3 \pmod{37}$, $1000000 \equiv 1 \pmod{37}$ and for 100 just a simple long division to get $100 \equiv 26 \pmod{37}$. Now we are left with $3 + x + 26y \equiv 0 \pmod{37}$. Note that if we know what is y, we can get the value of x easily. Now we try all possible values of $y = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ and the only ones that gives x a single digit number is $y = 1, 4, 7$, which gives the only solutions $(x, y) = (0, 7), (4, 4), (8, 1)$. □

6. (Junior 2013) Let $a < b < c < d < e$ be real numbers. Among the 10 sums of the pairs of these numbers, the least three are 32, 36 and 37 while the largest two are 48 and 51. Find all possible values of e.

   Proof. Write down all the 10 sums of pairs, and we see that the 2 smallest are $a + b = 32, a + c = 36$ and the 2 largest are $d + e = 51, c + e = 48$. Now the 3rd smallest can either be $b + c = 37$ or $a + d = 37$.

   <u>Case 1:</u> $b + c = 37$.
   We know $a + b$ and $a + c$, so solving we get $a = 15.5, b = 16.5, c = 20.5$, and from there we can get $e = 27.5, d = 23.5$, and this is a solution.

   <u>Case 2:</u> $a + d = 37$.
   Note that $a+c+d+e = (a+d)+(c+e) = 37+48 = 85$, but also $a+c+d+e = (a+c)+(d+e) = 36+51 = 87$, a contradiction.

   So the only possible value of e is 27.5. □

7. (Junior 2013) Six musicians gathered at a chamber music festival. At each scheduled concert some of the musicians played while the others listened as members of the audience. What is the least number of such concerts which would need to be scheduled so that for every two musicians each must play for the other in some concert.

   Proof. Let the musicians be A, B, C, D, E, F. 4 concerts is possible: in each concert 3 will be playing as follows: $(ABC), (ADE), (BDF), (CEF)$. It can be checked that for every two musicians each must play for the other in some concert. Now we show that it is impossible to do so with 3 concerts. Note that everybody has to perform at least once, so there is a concert with at least 2 performers, say A, B performed in concert 1. A has to listen to B and vice versa is some other concert, so we must have say A perform while B listen in concert 2, and B perform and A listen in concert 3. But now A is only listening to concert 3, and he must listen to everybody. So B, C, D, E, F must perform at concert 3. Similarly A, C, D, E, F must perform at concert 2. But now it is impossible for C, D to play and listen to each other. So 4 is the least number of concerts needed. □

8. (Junior 2016) A group of tourists get on 10 buses in the outgoing trip. The same group of tourists get on 8 buses in the return trip. Assuming each bus carries at least 1 tourist, prove that there are at least 3 tourists such that each of them has taken a bus in the return trip that has more people than the bus he has taken in the outgoing trip.

   Proof. Suppose there are n tourists. Let a_i, b_i be the number of tourists in the outgoing and return buses respectively for tourist i (including himself). Now we want to find 3 different i's such that $b_i > a_i$. Note that for each bus, the sum of $\frac{1}{a_i}$ over everyone in that bus is exactly 1. So the sum $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 10$ the number of buses. Similarly $\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} = 8$. So $\left(\frac{1}{a_1} - \frac{1}{b_1}\right) + \left(\frac{1}{a_2} - \frac{1}{b_2}\right) + \cdots + \left(\frac{1}{a_n} - \frac{1}{b_n}\right) = 2$. But $\left(\frac{1}{a_i} - \frac{1}{b_i}\right) < 1$ for each i, so there must be at least 3 different i's such that $\left(\frac{1}{a_i} - \frac{1}{b_i}\right)$ is positive. But that is the same as $b_i > a_i$ and we are done. □