Ambiguous points

Lim Jeck

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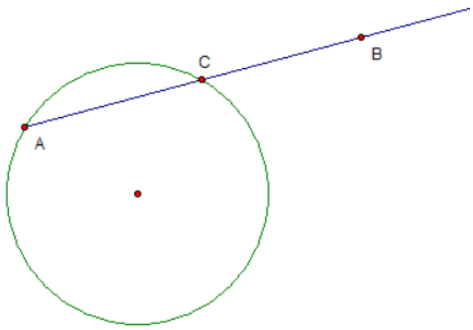
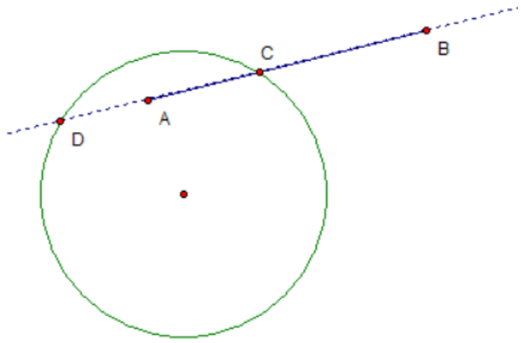
1 Ambiguous points

Let Γ be a circle and A, B be two points. If A lies on Γ, saying C is the intersection of AB with Γ with $C \neq A$ is unambiguous. However, if A does not lie on Γ, even if the segment AB intersects Γ only once, saying C is the intersection of AB with Γ is ambiguous. Algebraically, there is no difference between the two intersections C, D.



(a) Unambiguous



(b) Ambiguous

If a problem contains ambiguous points, it is usually useful to consider the other point of intersection, D. This special point usually has some special properties, and you will most likely construct it in a different way from AB intersect Γ. Most problems do not have ambiguous points, though. However, some can be rephrased in a way that involves ambiguous points.

2 Examples

1. (Steiner) Let Γ be the circumcircle of $\triangle ABC$ with orthocenter H. Let l be a line through H. Prove that the reflections of l about AB, BC, CA intersect at a point on Γ.

2. (ISL 2003) Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

3 Problems

1. (ARMO 2018) Let $\triangle ABC$ be an acute-angled triangle with $AB < AC$. Let M and N be the midpoints of AB and AC, respectively; let AD be an altitude in this triangle. A point K is chosen on the segment MN so that $BK = CK$. The ray KD meets the circumcircle Ω of ABC at Q. Prove that C, N, K, Q are concyclic.

2. (Serbia 2010) In an acute-angled triangle ABC, M is the midpoint of side BC, and D, E and F the feet of the altitudes from A, B and C, respectively. Let H be the orthocenter of ΔABC, S the midpoint of AH, and G the intersection of FE and AH. If N is the intersection of the median AM and the circumcircle of ΔBCH, prove that $\angle HMA = \angle GNS$.

3. Let $ABCD$ be a quadrilateral inscribed in a circle k. AC and BD meet at E. The rays CB, DA meet at F. Prove that the line through the incenters of $\triangle ABE, \triangle ABF$ and the line through the incenters of $\triangle CDE, \triangle CDF$ meet at a point lying on the circle k.

4. (ISL 2004) Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear, where $E = AC \cap BD$ and $F = BC \cap DA$.

5. (IMO 2019) Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

   Prove that lines DI and PQ meet on the line through A perpendicular to AI.