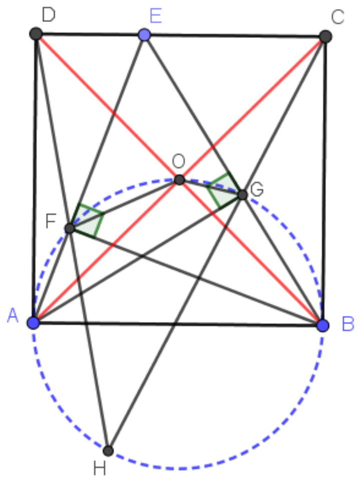
SMO Senior 2023 Rd.2

1. Let $ABCD$ be a square, E be a point on the side DC, F and G be the feet of the altitudes from B to AE and from A to BE, respectively. Suppose DF and CG intersect at H. Prove that $\angle AHB = 90°$.

Solution:



Let AC, BD intersect at O. Since $ABCD$ is a square, $\angle AOB = 90°$. Hence A, F, O, G, B lie on the circle with diameter AB.

Note that $\angle EFO = \angle OBA = 45° = \angle EDO$, so D, E, O, F are concyclic. Similarly, $\angle EGO = \angle OAB = 45° = \angle ECO$, so E, C, O, G are concyclic.

Thus $\angle FOG = 360° - \angle EOF - \angle EOG = (180° - \angle EOF) + (180° - \angle EOG) = \angle EDF + \angle ECG = 180° - \angle FHG$, so F, O, G, H are concyclic. This implies H also lies on the circle with diameter AB, so $\angle AHB = 90°$.

Remarks:

1) One might note that O is a Miquel Point in ΔHCD after finding the two cyclic quadrilaterals $DEOF$ and $ECOG$.
2) Coordinate Geometry is also rather suitable for this problem, since $ABCD$ is a square.

2. Find all positive integers k such that there exist positive integers a, b such that
$$a^2 + 4 = (k^2 - 4)b^2.$$

Solution:

We claim that only $k = 3$ works. It is clear that for $k = 3$, $a = b = 1$ will satisfy the equation.

Since $a^2 < (k^2 - 4)b^2 < k^2b^2$, let $a = kb - c$ where c is a positive integer. Then substituting and simplifying, we get

$$k = \frac{4b^2 + 4 + c^2}{2bc}$$

Clearly, this implies c is even. So letting $c = 2x$, we get:

$$k = \frac{b^2 + x^2 + 1}{bx}.$$

(*Remark: It is a "well-known", but hard-to-prove fact that the only possible integer value of the RHS expression is 3. The proof uses Vieta jumping, and is reproduced below.)*

Suppose there exists a $k \neq 3$ with such b, x. Let (b, x) be the pair with minimal sum such that $\frac{b^2+x^2+1}{bx} = k$. WLOG assume $x \geq b > 0$.

Then $x^2 - (kb)x + (b^2 + 1) = 0$.

Let x' be the 2nd root to this quadratic. By Vieta's,

$$x + x' = kb$$
$$xx' = b^2 + 1$$

Since $x' = kb - x, x'$ is an integer. Since $x' = \frac{b^2+1}{x}$, x' is positive. Hence, (b, x') is another pair which satisfies the equation, by the minimality assumption, $x' \geq x$.

Then $b^2 + 1 = xx' \geq x^2 \geq b^2$.

Clearly, $b^2 + 1$ is not a square. Thus, $x^2 = b^2$, implying that $x = b$.

Substituting back, we get $k = \frac{b^2+b^2+1}{b^2} = 2 + \frac{1}{b^2}$, which implies that the only integer combination is $b = 1, k = 3$, contradicting $k \neq 3$.

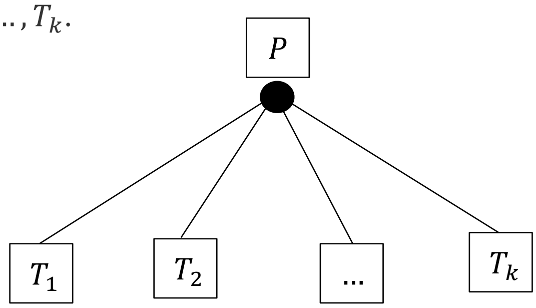
3. Let n be a positive integer. There are n islands with $n-1$ bridges connecting them such that one can travel from any island to another. One afternoon, a fire breaks out in one of the islands. Every morning, it spreads to all neighbouring islands. (Two islands are neighbours if they are connected by a bridge.) To control the spread, one bridge is destroyed every night until the fire has nowhere to spread the next day. Let X be the minimum possible number of bridges one has to destroy before the fire stops spreading. Find the maximum possible value of X over all possible configurations of bridges and islands where the fire starts at.

Solution:

We claim that the maximum possible value of X is $\lfloor\sqrt{n-1}\rfloor$.

We use a graph representation, where the islands are vertices and bridges are edges. Moreover, it is a connected graph with $n-1$ edges and n vertices, which implies it must be a tree (i.e. have no cycles).

We may represent it as a *rooted* tree where the root node P is the location where the fire starts at, and P is connected to the subtrees $T_1, T_2, \dots, T_k$.



Construction: For any positive integer k, consider the case with $n = k^2 + 1$ vertices, and each of $T_1, T_2, \dots, T_k$ is a line graph of k vertices. Clearly, the fire will spread along each T_i for up to k turns until a bridge is destroyed. After $k-1$ bridges are destroyed, some tree T_i would not have had any bridges destroyed. Thus, $X = k$ in this case. For any $k^2 + 1 < n \leq (k+1)^2$, we may simply add more vertices to T_k and the argument still holds. This shows that $X = \lfloor\sqrt{n-1}\rfloor$ is attainable.

Optimality:

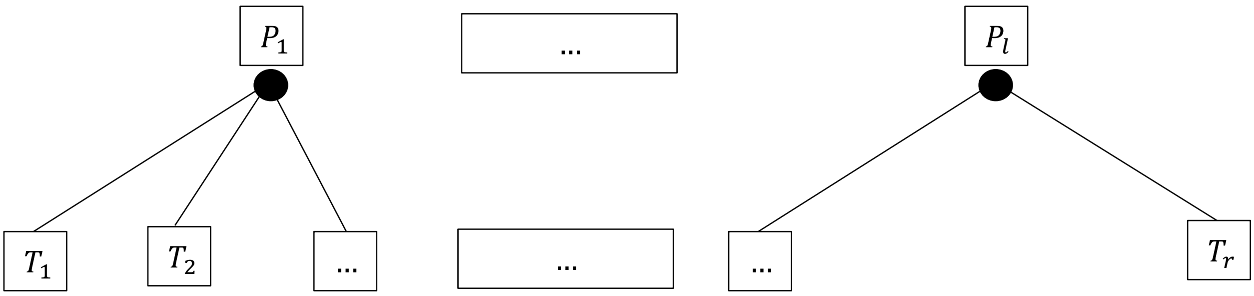
Without loss of generality, one may assume that an island on fire disappears after spreading to all its neighbouring islands. The configuration can then be viewed as a forest (i.e. a collection of trees) with their root nodes on fire; on each turn, each tree's root node is removed and its adjacent nodes become root nodes (on fire) of the respective subtrees.

At any point in time, call an island *fireable* if it is not yet on fire, but there exists a path from another island on fire to it.

Claim: If there are m *fireable* islands just before the time to destroy a bridge, then it suffices to destroy $\lfloor\sqrt{m}\rfloor$ bridges to stop the fire from spreading.

Proof (by strong induction on m): Consider a configuration with $k^2 \leq m < (k+1)^2$ *fireable* islands, and we aim to destroy at most k bridges.

Let the configuration contain l trees with root nodes $P_1, P_2, \ldots, P_l$ on fire, and suppose they are connected to subtrees $T_1, T_2, \ldots, T_r$ with $n_1, n_2, \ldots, n_r$ vertices respectively.



WLOG let $n_1 \leq n_2 \leq \cdots \leq n_r$. By definition, $n_1 + n_2 + \cdots + n_r = m$.

Moreover, if any $n_i \leq k$, we can 'ignore' the subtree T_i entirely, since the fire would definitely have finished spreading within k days. Thus, WLOG we may assume all trees have $\geq k+1$ vertices. Then since $m < (k+1)^2$, $r \leq k$. Thus, $n_r \geq \left\lceil \frac{m}{r} \right\rceil$.

We now destroy the bridge from P_l to the root node of T_r. The next morning, we 'remove' T_r, and all other root nodes are on fire. We will be left with $m-(r-1)-n_r \leq m+1-r-\frac{m}{r} \leq (m+1)-2\sqrt{m}\ (\text{AM}-\text{GM}) = \left(\sqrt{m}-1\right)^2$ fireable nodes. Moreover, equality only can hold if m is a perfect square, i.e. $m=k^2$, and $r=\frac{m}{r}$, i.e. $k=r$. However, if $m=k^2, n_1 \geq k+1 \Rightarrow r \leq \left\lfloor \frac{k^2}{k+1} \right\rfloor = k-1$, thus equality cannot hold. So there will be strictly less than $\left(\sqrt{m}-1\right)^2$ fireable nodes. By the induction hypothesis, destroying a further $\lfloor\sqrt{m}\rfloor - 2$ bridges is sufficient for these, so we have stopped the fire by destroying $\lfloor\sqrt{m}\rfloor - 1$ bridges as desired.

Applying this claim to the general case gives the desired conclusion, since there are $n-1$ *fireable* islands at first.

Remarks:

1) The construction is not too difficult here, the surprising thing is that it works! The main idea is the notion that we can let small subtrees "burn up quickly", so the worst case would be multiple medium-sized subtrees.
2) The proof is much clearer on a specific case (e.g. $n = 100$); the intuition is slightly obfuscated by the number of variables used.

4. Find all positive integers m, n satisfying $n! + 2^{n-1} = 2^m$.

Solution:

We claim $(m, n) = (1,1), (2,2), (5,4)$ are the only solutions. Suppose there are other solutions. Let $v_2(n)$ denote the largest k such that 2^k divides n. Observe that $m = v_2(RHS) = v_2(LHS)$.

Additionally, by Legendre's formula, $v_2(n!) = n - s_2(n)$, where $s_2(n)$ is the number of 1s in the binary representation of n.

We now consider two cases.

Case 1: n is not a power of 2.

Then $s_2(n) > 1$, so $v_2(n!) \leq n - 2$. Thus, $m = v_2(n! + 2^{n-1}) = v_2(n!)$. This clearly implies $2^m \leq n! < LHS$, contradiction.

Case 2: $n = 2^k$ for some $k \geq 3$.

Then $v_2(n!) = n - 1 = v_2(2^{n-1})$, so we may factor the LHS as $2^{n-1}\left(\frac{n!}{2^{n-1}} + 1\right)$. Removing all factors of 2 from $1,2,3, \dots, n$ allows us to write $\frac{n!}{2^{n-1}}$ as $(1) \cdot (1 \cdot 3) \cdot (1 \cdot 3 \cdot 5 \cdot 7) \cdot \dots \left(1 \cdot 3 \cdot 5 \cdot \dots \cdot (2^k - 1)\right)$.

Note that since $1 \cdot 3 \cdot 5 \cdot 7 \equiv 1 \pmod 8$, this product is congruent to $3 \pmod 8$ for $k \geq 3$. Hence, $\frac{n!}{2^{n-1}} + 1 \equiv 4 \pmod 8$, so $v_2\left(2^{n-1}\left(\frac{n!}{2^{n-1}} + 1\right)\right) = (n-1) + 2 = n + 1$.

Thus, $m = n + 1$, which gives $n! + 2^{n-1} = 2^{n+1}$.

But then $n! = 2^{n+1} - 2^{n-1} = 3 \cdot 2^{n-1}$, which clearly fails since $5 | n!$ for $n \geq 2^3$.

Therefore, there are indeed no other solutions.

Remark: Legendre's Formula exists in 2 forms; the other is the more familiar $v_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$, which is commonly used to count zeroes at the end of $n!$.

5. Colour a 20000×20000 square grid using 2000 different colours with 1 colour in each square. Two squares are neighbours if they share a vertex. A path is a sequence of squares so that 2 successive squares are neighbours. Mark k of the squares. For each unmarked square x, there is exactly 1 marked square y of the same colour so that x and y are connected by a path of squares of the same colour. For any 2 marked squares of the same colour, any path connecting them must pass through squares of all the colours. Find the maximum value of k.

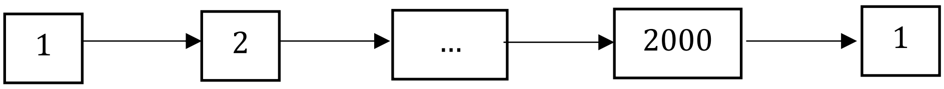
Solution:

We claim that the maximum value of k is simply 20000. This can be achieved by colouring entire rows with colours $1,2,3,\dots,2000$ and repeating this cycle 10 times, and then marking the first square of each row. Clearly, this satisfies the condition.

Proof of Optimality:

We may divide the square into connected *regions*, each consisting of squares of a single colour. The first condition implies that each region has exactly 1 marked square; moreover, the position of the marked square does not matter. Thus, we are actually aiming to maximise the number of regions.

To obtain $k > 20000$, obviously, some colour must appear in more than 1 region. WLOG suppose there are ≥ 2 regions of colour 1, and one of the paths between these regions passes through colours $2,3,4,\dots,2000$ in that order.



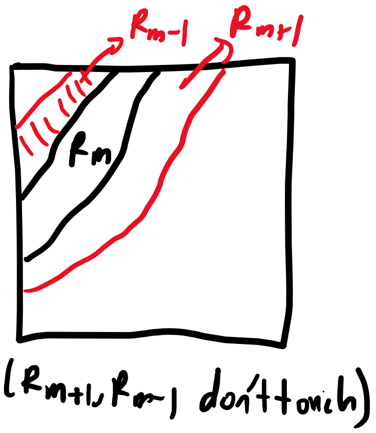
If any of the regions coloured $2,3,\dots,2000$ are connected to another additional region with colour k, then there clearly exists a path between the colour-k regions that does not pass through the colour 1. (If $k = 1$, then there is a path between colour-1 regions that does not pass through either colour 2 or 2000).

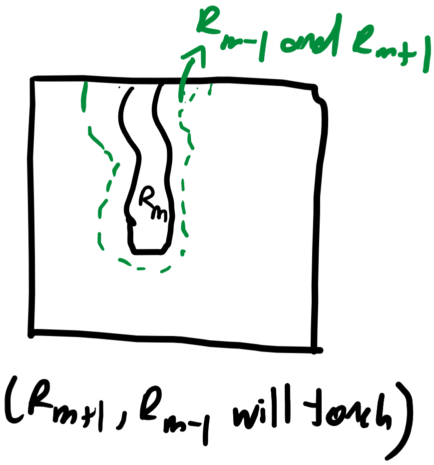
By this reasoning, it is easy to see that the regions can only touch each other in the order $1,2,3,\dots,2000,1,2,3,\dots,2000,\dots$. $(*)$

Now label these regions $R_1, R_2, \dots, R_n$ in order, and consider the shortest path between any square in R_1 and any square in R_n. Since it has to pass through $R_1, R_2, R_3, \dots, R_n$ in some order, this path has at least n squares. But clearly, any two squares have a path between them with at most 20000 squares. Hence $n \leq 20000$.

(∗) There is one potential case to consider – what if they touch each other in a cyclical fashion, ending in a colour-2000 region touching the first colour-1 region. (The shortest path argument would not quite work in that case.)

We claim, however, that this is not possible. Supposing by contradiction that it is possible, each region R_m must touch exactly 2 other regions R_{m+1} and R_{m-1}, which themselves do not touch. Observe that this is only possible if R_m 'disconnects' the square:





But then this implies that regions on the R_{m-1} "side" and R_{m+1} "side" will never have a path between them that doesn't pass through R_m, so a cycle is not possible.

Remark: Oddly enough, this is rather easy for Q5. However, the problem formulation is rather confusing, which makes it challenging to get started.