Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2023

Senior Section (Round 1)

Tuesday, 30 May 2023 **0930 – 1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Find the value of m such that $2x^2 + 3x + m$ has a minimum value of 9.

(A) $\frac{9}{8}$ (B) $-\frac{9}{8}$ (C) $\frac{81}{8}$ (D) $-\frac{81}{8}$ (E) $\frac{63}{8}$

2. Which of the following is true?

(A) $\sin(105°) - \cos(105°) = \frac{\sqrt{3}}{2}$ (B) $\sin(105°) - \cos(105°) = \frac{\sqrt{3}}{\sqrt{2}}$

(C) $\sin(105°) + \cos(105°) = \frac{1}{2}$ (D) $\sin(105°) + \cos(105°) = \frac{1}{\sqrt{3}}$

(E) None of the above.

3. If $\log_{\sqrt{2}} x = 10 - 3\log_{\sqrt{2}} 10$, find x.

(A) 0.32 (B) 0.032 (C) 0.0032 (D) 0.64 (E) 0.064

4. If $(x-5)^2 + (y-5)^2 = 5^2$, find the smallest value of $(x+5)^2 + (y+5)^2$.

(A) $225 - 100\sqrt{2}$ (B) $225 + 100\sqrt{2}$ (C) $225\sqrt{2}$ (D) $100\sqrt{2}$
(E) None of the above

5. Suppose $\cos(180° + x) = \frac{4}{5}$, where $90° < x < 180°$. Find $\tan(2x)$.

(A) $\frac{24}{7}$ (B) $\frac{7}{24}$ (C) $-\frac{24}{7}$ (D) $-\frac{7}{24}$ (E) $-\frac{24}{25}$

Short Questions

6. Suppose the roots of $x^2 + 11x + 3 = 0$ are p and q, and the roots of $x^2 + Bx - C = 0$ are $p + 1$ and $q + 1$. Find C.

7. If the smallest possible value of $(A - x)(23 - x)(A + x)(23 + x)$ is $-(48)^2$, find the value of A if $A > 0$.

8. Find the smallest positive odd integer greater than 1 that is a factor of

$$(2023)^{2023} + (2026)^{2026} + (2029)^{2029}.$$

9. Find the remainder of $7^{2023} + 9^{2023}$ when divided by 64.

10. Let $x, y, z > 1$, and let A be a positive number such that $\log_x A = 30$, $\log_y A = 50$ and $\log_{xy}(Az) = 150$. Find $(\log_A z)^2$.

11. Find the largest integer that is less than

$$\frac{3^{10}-2^{10}}{10!}\left(\frac{1}{1!9!2}+\frac{1}{2!8!2^2}+\frac{1}{3!7!2^3}+\cdots+\frac{1}{9!1!2^9}\right)^{-1}.$$

Here, $n! = n\cdot(n-1)\cdots 3\cdot 2\cdot 1$. For example, $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$.

12. Consider the following simultaneous equations:

$$\begin{aligned} xy^2 + xyz &= 91, \\ xyz - y^2z &= 72, \end{aligned}$$

where x, y, and z are positive integers. Find the maximum value of xz.

13. Let x be a real number such that

$$\frac{\sin^4 x + \cos^4 x}{\sin^2 x + \cos^4 x} = \frac{8}{11}.$$

Assuming $\sin^2 x > \frac{1}{2}$, find the value of $\sqrt{28}(\sin^4 x - \cos^4 x)$.

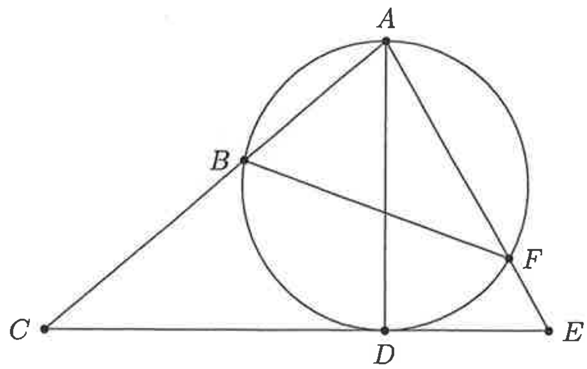
14. A sequence a_1, a_2, …, is defined by

$$a_1 = 5,\ a_2 = 7,\ a_{n+1} = \frac{a_n + 1}{a_{n-1}} \quad \text{for } n \geq 2.$$

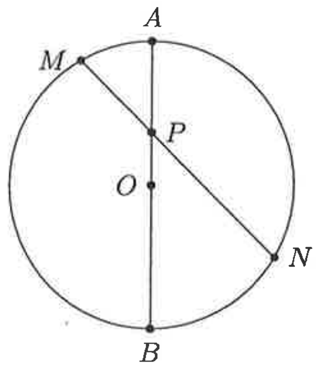
Find the value of $100 \times a_{2023}$.

15. Let C be a constant such that the equation $5\cos x + 6\sin x - C = 0$ have two distint roots a and b, where $0 < b < a < \pi$. Find the value of $61 \times \sin(a+b)$.

16. In the diagram below, CE is tangent to the circle at point D, AD is a diameter of the circle, and ABC, AFE are straight lines. It is given that $\frac{AB}{AC} = \frac{16}{41}$ and $\frac{AF}{AE} = \frac{49}{74}$. Let $\tan(\angle CAE) = \frac{m}{n}$, where m, n are positive integers and $\frac{m}{n}$ is a fraction in its lowest form. Find the sum $m + n$.



17. In the diagram below, AB is a diameter of the circle with centre O, MN is a chord of the circle that intersects AB at P, $\angle BON$ and $\angle MOA$ are acute angles, $\angle MPA = 45°$, $MP = \sqrt{56}$, and $NP = 12$. Find the radius of the circle.



18. Let $f(x) = \cos^2\left(\frac{\pi x}{2}\right)$. Find the value of

$$f\left(\frac{1}{2023}\right) + f\left(\frac{2}{2023}\right) + \cdots + f\left(\frac{2021}{2023}\right) + f\left(\frac{2022}{2023}\right).$$

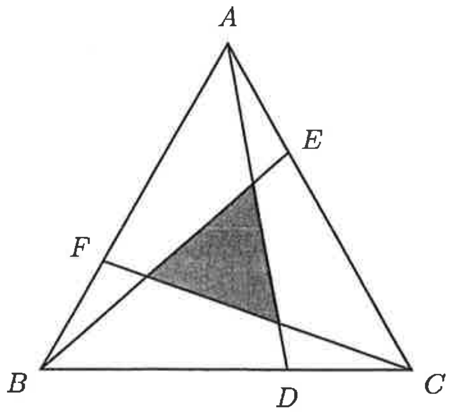
19. Find the remainder when 3^{2023} is divided by 215.

20. Find the sum of the prime divisors of 64000027.

21. Let $\triangle ABC$ be an equilateral triangle. D, E, F are points on the sides such that

$$BD : DC = CE : EA = AF : FB = 2 : 1.$$

Suppose the area of the triangle bounded by AD, BE and CF is 2023. Find the area of $\triangle ABC$.



22. Find the number of possible ways of arranging m ones and n zeros in a row such that

there are in total $2k+1$ strings of ones and zeros. For example,

$$1110001001110001$$

consists of 4 strings of ones and 3 strings of zeros.

23. Suppose that there exist numbers a, b, c and a function f such that for any real numbers x and y,

$$f(x+y)+f(x-y)=2f(x)+2f(y)+ax+by+c.$$

It is given that

$$f(2)=3, \quad f(3)=-5, \quad \text{and} \quad f(5)=7.$$

Find the value of $f(123)$.

24. Let f be a function such that for any nonzero number x,

$$6xf(x)+5x^2f(1/x)+10=0.$$

Find the value of $f(10)$.

25. Find the number of triangles such that all the sides are integers and the area equals the perimeter (in number).