Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2017

Senior Section (Round 1)

Tuesday, 30 May 2017 | 0930 – 1200 hrs

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

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Multiple Choice Questions

1. Find all the values of k for which the expression $x^2 + k(k-x) + 3k$ is positive for all x.

   (A) $0 < k < 4$ (B) $-4 < k < 0$ (C) $k < 0,\ k > 4$ (D) $k > 0,\ k < -4$

   (E) None of the above

2. Simplify

$$\log_3 12 + \log_9 16 - \log_{27} 8.$$

   (A) $\log_3 4$ (B) $\log_3 8$ (C) $\log_3 16$ (D) $\log_3 24$ (E) $\log_3 32$

3. Solve for x in the following equation

$$7^{x-1} + \frac{1}{7} - \frac{6}{7^x} = 0.$$

   (A) $\log_6 7$ (B) $\log_6 14$ (C) $\log_7 6$ (D) $\log_7 12$ (E) $\log_7 14$

4. When a polynomial is divided by $(x-2)$ and $x^2 - 3x + 2$, the remainders are 11 and $Ax + 5$ respectively, where A is some integer. Find the remainder when it is divided by $(x-1)$.

   (A) 7 (B) 8 (C) 9 (D) 10 (E) 11

5. Simplify

$$\frac{\sqrt{5}^3 - \sqrt{2}^3}{\sqrt{5} - \sqrt{2}} - \frac{\sqrt{5}^3 + \sqrt{2}^3}{\sqrt{5} + \sqrt{2}}.$$

   (A) $\sqrt{10}$ (B) $2\sqrt{10}$ (C) $3\sqrt{10}$ (D) $\sqrt{5} - \sqrt{2}$ (E) $\sqrt{5} + \sqrt{2}$

6. Which of the following is the largest?

   (A) 3^{100} (B) 90^{10} (C) 82^{25} (D) $8^{49} + 7^{49}$ (E) 2^{150}

7. Suppose $270° < a < 360°$. Which of the following is equal to

$$\sqrt{\cos^3 a} - \sqrt{\sin^2 a \cos a}\ ?$$

   (A) $(\sin a + \cos a)\sqrt{\cos a}$ (B) $(\sin a - \cos a)\sqrt{\cos a}$ (C) $(\cos a - \sin a)\sqrt{\cos a}$

   (D) $(-\sin a - \cos a)\sqrt{\cos a}$ (E) $\sqrt{\cos a}$

8. Find the range of values of x that satisfy both of the following inequalities:

$$(x+3)^2 \leq (3x-1)^2, \quad 3x \geq x^2.$$

(A) $0 \leq x \leq 1$ (B) $1 \leq x \leq 2$ (C) $2 \leq x \leq 3$ (D) $3 \leq x \leq 4$
(E) None of the above

9: If $f(x) = -x + 6\sqrt{x+16} - 5$, where $-16 \leq x \leq 0$, find the range of $f(x)$.

(A) $-29 \leq f(x) \leq 20$ (B) $-29 \leq f(x) \leq 19$ (C) $11 \leq f(x) \leq 19$

(D) $19 \leq f(x) \leq 20$ (E) $11 \leq f(x) \leq 20$

10. A circle is given by the equation $x^2 - 4x + y^2 + 8y = 5$. Find the equation of the tangent line to the circle at the point $(-2, -1)$.

(A) $3y - 4x - 5 = 0$ (B) $-7y + x - 5 = 0$ (C) $3y + 2x + 7 = 0$

(D) $3y - 2x - 1 = 0$ (E) $4y - 3x - 2 = 0$

Short Questions

11. Suppose $x^2 + 3x + 15$ is a factor of $2x^4 + mx^2 + 30n$ where m, n are integers. Find the value of mn.

12. How many different real numbers x, where $0° \leq x \leq 360°$, satisfy the equation

$$3\sin x - 4\sin^3 x = \frac{1}{2} \ ?$$

13. Suppose $\dfrac{1-\sin 2A}{1+\cos 2A} = \tan A - 1 \neq 0$. Find $\tan^2 A$.

14. Given that $\sin\left(\dfrac{\pi}{4} - x\right) = \dfrac{3}{5}$, $0 < x < \dfrac{\pi}{4}$, find $\dfrac{15\cos 2x}{\cos\left(\frac{\pi}{4} + x\right)}$.

15. Find the coefficient of x^3 in the expansion of $(1 + 2x + 3x^2 + 4x^3)^8$.

16. Find the maximum value of $\dfrac{1 + x - 99x^2}{x(x^2+1)}$, where $\dfrac{1}{100} \leq x \leq \dfrac{1}{10}$.

17. Four points O, P, Q, R have coordinates $(0,0)$, $(0,30)$, (x,y) and $(20,0)$ respectively, where the point Q lies on the curve $y = 50 - x^2$ in the first quadrant. Let Π be the polygon $OPQR$ with four sides OP, PQ, QR and RO. What is the maximum possible area of Π if this area must be an integer?

18. There are 5 girls and 5 boys in a junior class, and 4 girls and 9 boys in a senior class. A committee of 7 members is to be formed by selecting students from these two classes. Find the number of ways this can be done if the committee must have exactly 4 seniors and exactly 5 boys.

19. Suppose x is a real number. Find the largest possible integer that can be attained by the expression

$$\frac{7770 - |x - 10|}{|x - 5| + |x - 15|}.$$

20. Evaluate

$$\left(10 - \frac{4\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right)\cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{2\pi}{7}\right) - 1}{2\cos\left(\frac{\pi}{7}\right)}\right)^2.$$

21. What is the last digit of 2017^{2017}?

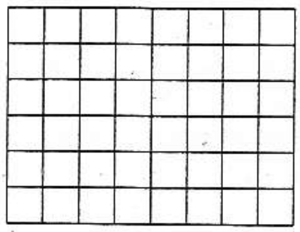
22. In how many ways can 4 integers, $a_1 < a_2 < a_3 < a_4$, be chosen from the integers 1, 2, 3, ..., 26 such that $5 \leq a_i - a_{i-1} \leq 7$ for all $i = 2, 3, 4$?

23. Let α and β be the roots of the quadratic equation $x^2 - 7x + 11 = 0$. Determine the value of $\alpha^4 + \beta^4$.

24. Let $\{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers. Let $\{b_1, b_2, b_3, \ldots\}$ be a sequence of real numbers such that $b_n = a_{n+1} - a_n$ for all $n \geq 1$. Determine the maximum value of a_n within the sequence $\{a_1, a_2, a_3, \ldots\}$ if $b_{m+1} - b_m = -2$ for all $m \geq 1$, $a_5 = 615$ and $a_{10} = 1045$.

25. There are 12 blue socks, 14 red socks, 16 green socks, 18 yellow socks and 20 orange socks in a drawer. Socks of the same colour are indistinguishable. A person randomly picks a certain number of socks from the drawer. Find the minimum number of socks that should be taken to ensure that he will have at least two pairs of colour X, at least two pairs of colour Y and at least two pairs of colour Z, for some three distinct colours X, Y, Z.

26. Determine the number of paths to move from the top-left cell to the bottom-right cell in the 8×6 cell grid using a sequence of downwards moves and rightwards moves such that there are an even number of direction changes.

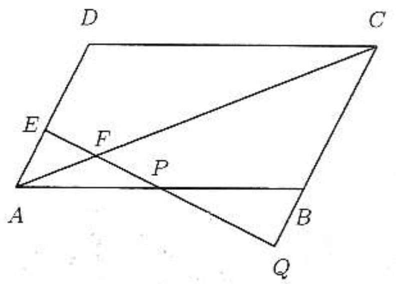


27. Consider the following equation where m and n are positive integers:

$$3^m + 3^n - 8m - 4n! = 680.$$

Determine the sum of all possible values of m.

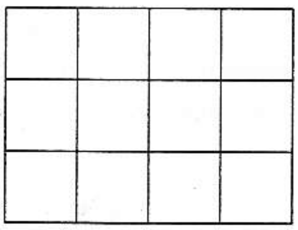
28. In the figure below, $ABCD$ is a parallelogram, where B lies on the straight line CQ. The line EQ intersects the lines AC and AB at the point F and P respectively, where P is the midpoint of the line AB. Let $|AE|$, $|AD|$, $|AC|$ and $|AF|$ denote the length of the line segment AE, AD, AC and AF respectively. If $|AE| = 3$ cm, $|AD| = 9$ cm, find $\dfrac{6 \times |AC|}{|AF|}$.



29. Find the smallest positive integer n such that

$$5(3^2 + 2^2)(3^4 + 2^4)(3^8 + 2^8)\cdots(3^{2^n} + 2^{2^n}) > 9^{256}.$$

30. The following diagram shows paths (edges in the grid) connecting 5×4 lattice points. Each path is exactly 1 meter long. Determine the shortest distance (in meters) a person needs to travel so that he will walk through each path at least once and returns to the starting position.



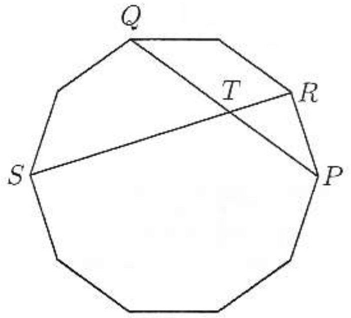
31. Let a, b, x and y be real variables such that

$$ax + by = 40, \quad ax^2 + by^2 = 110, \quad ax^3 + by^3 = 310, \quad ax^4 + by^4 = 890.$$

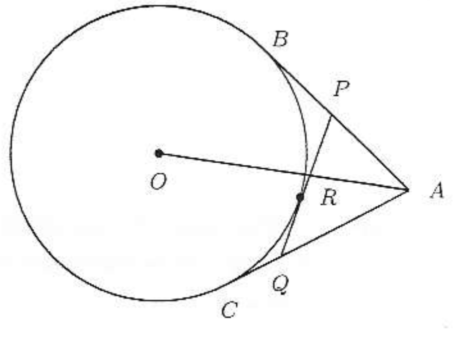
Determine the value of $ax^5 + by^5$.

32. Determine the largest prime number that cannot be expressed as the sum of three composite odd integers.

33. Two lines PQ and RS connect the vertices of a regular decagon (10-sided polygon) and intersect at the point T. Suppose $\angle PTS = x^\circ$. Find x.



34. In the figure below, the lines AB, AC and PQ are tangent to the circle (with centre O) at the points B, C and R respectively. If the circle has radius 9 cm and the length of the line segment OA is 15 cm, find the perimeter of $\triangle APQ$ in cm.



35. In the figure below, the circle centred at A has radius 20 cm and the circle centred at B has radius 5 cm. The line segment QR is tangent to the smaller circle at the point P. Let $|QB|$ and $|BR|$ denote the length of the line segment QB and BR respectively. Find the value of $|QB| \times |BR|$.