Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2016

Senior Section (Round 1)

Tuesday, 31 May 2016 — 0930 – 1200 hrs

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, the constant* e *is the base of the natural logarithm* ln.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

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Multiple Choice Questions

1. If α, β are two distinct roots of the equation $2x^2 + 4x - 45 = 0$, find the value of $\alpha^2\beta + \alpha\beta^2$.

   (A) 15 (B) -15 (C) 45 (D) -45 (E) 90

2. Simplify

$$\frac{9^{\frac{8}{3}} - 9^{\frac{5}{3}}}{4^{\frac{8}{3}} - 4^{\frac{2}{3}}}.$$

   (A) $\dfrac{3^{\frac{7}{3}}5}{2^{\frac{5}{3}}}$ (B) $3^{\frac{7}{3}}2^{\frac{5}{3}}$ (C) $\dfrac{3^{\frac{5}{3}}2^{\frac{5}{3}}}{5}$ (D) $\dfrac{3^{\frac{7}{3}}2^{\frac{5}{3}}}{5}$ (E) $\dfrac{3^{\frac{4}{3}}2^{\frac{1}{3}}}{5}$

3. Given that $e^{2x}e^{y} = e^2$ and $\ln(x + 2y) = \ln 5 + \ln 2$, find the value of $x + y$.

   (A) 2 (B) 4 (C) 8 (D) 10 (E) 12

4. Solve for x in the following equation

$$\log_9 x + \log_3(27x) = 30.$$

   (A) 3^{16} (B) 3^{17} (C) 3^{18} (D) 3^{19} (E) 3^{20}

5. Given that $\cos\alpha = \frac{12}{13}$, where $0^\circ \le \alpha \le 90^\circ$, and that $\tan\beta = \frac{3}{4}$, where $180^\circ \le \beta \le 270^\circ$. Find $\sin(\beta - \alpha)$.

   (A) $-\dfrac{16}{65}$ (B) $\dfrac{16}{65}$ (C) $-\dfrac{56}{65}$ (D) $\dfrac{56}{65}$ (E) $\dfrac{33}{65}$

6. Which of the following is the greatest?

   (A) $\sqrt{\dfrac{1}{6}}$ (B) $\left(\dfrac{1}{218}\right)^{1/6}$ (C) $\left(\dfrac{1}{15}\right)^{1/3}$ (D) $\left(\dfrac{1}{7}\right)^{1/3}\left(\dfrac{1}{5}\right)^{1/6}$ (E) $\left(\dfrac{1}{8}\right)^{1/4}\left(\dfrac{1}{3}\right)^{1/3}$

7. Find all the positive values of x for which

$$2|x^2 - 3x| \le x.$$

   (A) $\dfrac{5}{2} \le x \le 3$ (B) $\dfrac{1}{2} \le x \le \dfrac{5}{2}$ (C) $\dfrac{5}{2} \le x \le \dfrac{7}{2}$ (D) $0 \le x \le \dfrac{7}{2}$ (E) $\dfrac{1}{2} \le x \le 1$

8. Suppose $0^\circ \le x \le 180^\circ$. Find all the possible values of x such that

$$3\cos^2 2x + 4\sin 2x - \sin^2 2x = 0.$$

(A) $x = 55°,\ 135°$ (B) $x = 75°,\ 125°$ (C) $x = 105°,\ 165°$ (D) $x = 95°,\ 145°$
(E) $x = 115°, 175°$

9. Which of the following is equal to

$$\frac{1}{\sqrt{2000}+\sqrt{2004}}+\frac{1}{\sqrt{2004}+\sqrt{2008}}+\frac{1}{\sqrt{2008}+\sqrt{2012}}+\frac{1}{\sqrt{2012}+\sqrt{2016}}\ ?$$

(A) $3\sqrt{14}-5\sqrt{5}$ (B) $4\sqrt{14}-5\sqrt{5}$ (C) $3\sqrt{14}-5\sqrt{3}$ (D) $3\sqrt{14}-3\sqrt{5}$

(E) $3\sqrt{7}-\sqrt{5}$

10. Let $a = \cos 282°$, $b = \cos 349°$, $c = \sin 102°$ and $d = \sin 169°$. Which of the following is true?

(A) $b > a > c > d$ (B) $b > c > a > d$ (C) $c > a > d > b$ (D) $a > d > b > c$

(E) $a > c > d > b$

Short Questions

11. The expression $3x^3 + Ax^2 + Bx - 10$, where A and B are integers, is divisible by $3x - 1$ but leaves a remainder of -14 when divided by $x + 2$. Find the value of $A + B$.

12. Find the largest positive integer p such that $-x^2 + 2(1 + 2p)x - 2p - 31$ is always negative.

13. Find the largest integer smaller than $(2 + \sqrt{2})^4$.

14. Given that $x > 0$, and $5^x + \frac{1}{5^x} = \frac{5}{2}$, find the value of $2^{\frac{2}{x}}$.

15. Find the largest value of x satisfying the following equation:

$$\sqrt{3x-47}+\sqrt{x-1}=6.$$

16. Find the number of solutions for the following equation:

$$|2\sin x + 1| = 2\,|\cos x|\,,$$

where $0° \leq x \leq 360°$.

17. Find the maximum value of $6\cos^2 x - 24\sin x\cos x - 4\sin^2 x$, where $0° \leq x \leq 360°$.

18. Given that $\tan\alpha = \frac{1}{3}$ and $\tan\beta = 3$. Find the value of

$$\frac{3\sin(\alpha+\beta) - 6\sin\alpha\cos\beta}{2\sin\alpha\sin\beta + \cos(\alpha+\beta)}.$$

19. Consider the function

$$f(x) = 3|x+2| + 3|x| - |x-1| - 5|x-2|,$$

where x is any real number. Suppose A and B are the maximum and minimum values of $f(x)$ respectively. Find the value of $A - B$.

20. Find the coefficient of x^4 in the expansion of

$$\left(x^2 + \frac{2}{x^2}\right)\left(x + \frac{1}{x^3}\right)^{10}.$$

21. Find the smallest integer $k > 23$ such that

$$\sqrt{k^2 - 23k}$$

is a positive integer.

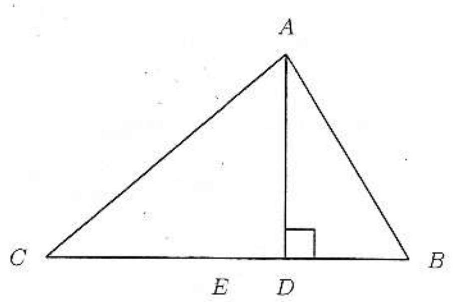
22. Suppose x and y are two real numbers such that

$$x + y = 6 \quad \text{and} \quad 2x^2 + 3xy + y^2 = 12.$$

Find the value of $x^2 + y^2$.

23. Suppose a circle C is centred at the point $(3,1)$ on the xy-plane, and the line $4y + 3x = 63$ is tangent to the circle C. Find the radius of C.

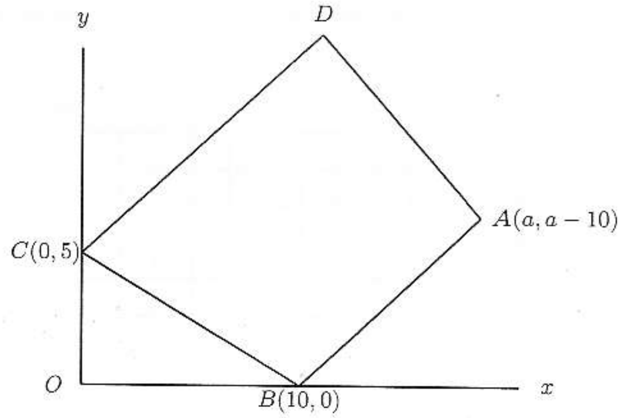
24. In the triangle ABC below, $\angle ABC = 2\angle ACB$, and AD is perpendicular to BC. Suppose E is the midpoint of BC, and $DE = 3$ meters. Find the length of AB in meter.



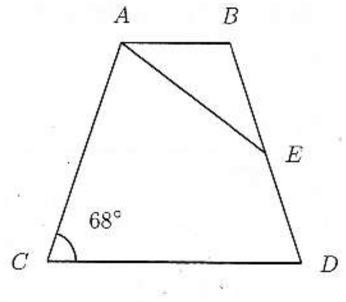
25. Suppose x and y are positive real numbers such that $1 \leq x + y \leq 9$ and $x \leq 2y \leq 3x$. Find the largest value of

$$\frac{9 - y}{9 - x}.$$

26. The figure below shows a quadrilateral $ABCD$ on the xy-plane, where $A(a, a - 10)$, $B(10, 0)$ and $C(0, 5)$. The line AB is parallel to the line CD, and the line AD is perpendicular to CD. Given that a is a positive integer greater than 10, find the smallest value of a such that the area of the quadrilateral $ABCD$ is greater than 200.



27. In the figure below, AB is parallel to CD, $AC = AB + CD$ and E is the midpoint of BD. Suppose $\angle ACD = 68°$, find the angle $\angle CAE$ in degree.



28. Suppose $\overline{abcabc}$ is the smallest 6-digit number which is divisible by 2016, where the digits a, b and c need not be distinct. Find the 3-digit number $\overline{abc}$.

29. Find the largest positive integer n such that 19^n divides $2016!$.

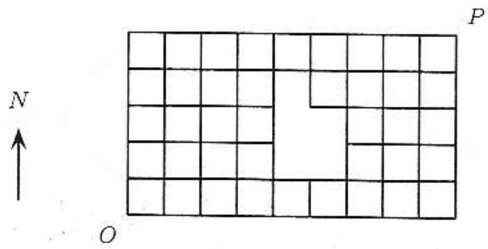
30. A sequence a_1, a_2, a_3, ... satisfies

$$a_{n+3} = 2a_{n+2} - 2a_{n+1} + a_n$$

for $n = 1, 2, 3, \ldots$. If $a_1 = 11$, $a_2 = 222$ and $a_3 = 7777$, find a_{2016}.

31. How many ways are there to arrange the letters of the word 'RECURRENCE' in a row so that no two R's are adjacent?

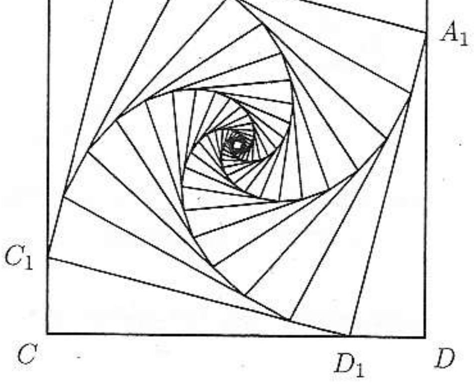
32. The roads of a town form a north-by-east grid as shown in the figure below. Find the number of ways a vehicle can go from point O to point P where only easterly and northerly directions are allowed and no backtrackings are allowed.



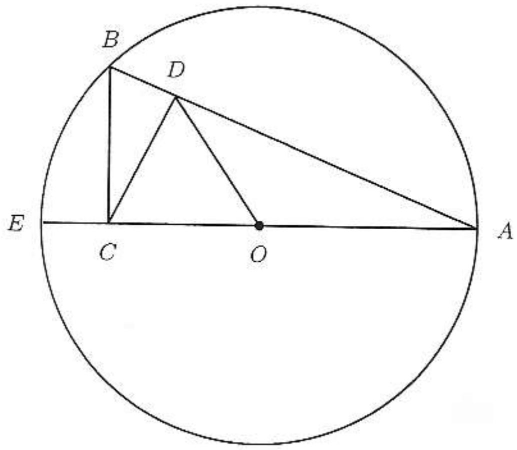
33. In the following diagram, $ABCD$ is a square with $AB = 10$ cm. Let A_1, B_1, C_1, D_1 be points on the sides of $ABCD$ such that $AA_1 = BB_1 = CC_1 = DD_1 = \frac{1}{6}AB$. Similarly, let A_2, B_2, C_2, D_2 be points on the sides of $A_1B_1C_1D_1$ such that $A_1A_2 = B_1B_2 = C_1C_2 = D_1D_2 = \frac{1}{6}A_1B_1$. Repeat this procedure to construct infinitely many squares

$$ABCD,\ A_1B_1C_1D_1,\ A_2B_2C_2D_2,\ \ldots.$$

Find the sum of their areas in cm^2.



34. In the figure below, the point O is the centre of the circle, the line BC is perpendicular to the line AE, and the line CD is perpendicular to the line AB. If the radius of the circle is 10 cm, find the value of $OD^2 + CD^2$ in cm^2.



35. Let $a_1, a_2, \ldots, a_{10}$ be an arrangement of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 such that

(i) $a_1 + a_2 + a_3 = a_4 + a_5 + a_6 = a_7 + a_8 + a_9$, and

(ii) the number a_{10} is even and not equal to 10.

How many such arrangements are there?