Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2024
(Open Section, Round 1)

Wednesday, 30 May 2024 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

In this paper, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x, and $\lceil x \rceil$ denote the smallest integer not less than x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, $\lfloor -2.3 \rfloor = -3$; $\lceil 5 \rceil = 5$, $\lceil 2.8 \rceil = 3$, $\lceil -2.3 \rceil = -2$

1. Let $S_k = 1 + 2 + 3 + \cdots + k$ for any positive integer k. Find $S_1 + S_2 + S_3 + \cdots + S_{20}$.

2. Let $S = \sum_{r=1}^{64} r\binom{64}{r}$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $0! = 1$. Find $\log_2 S$

3. Let x be the largest number in the interval $[0, 2\pi]$ such that $(\sin x)^{2024} - (\cos x)^{2024} = 1$. Find $\lfloor x \rfloor$.

   (Note: If you think that such a number x does not exist, enter your answer "99999".)

4. Find the number of real numbers x that satisfies the equation $|x - 2| + |x - 3| = |2x - 5|$.

   (Note: If you think that there are no such numbers, enter "0"; if you think that there are infinitely many such numbers, enter "99999".)

5. Among all the real numbers that satisfies the inequality $e^x \geq 1 + 2e^{-x}$, find the minimum value of $\lceil e^x + e^{-x} \rceil$.

6. Find the smallest positive integer C greater than 2024 such that the sets $A = \{2x^2 + 2x + C : x \in \mathbb{Z}\}$ and $B = \{x^2 + 2024x + 2 : x \in \mathbb{Z}\}$ are disjoint.

7. Let $ABCD$ be a convex quadrilateral inscribed in a circle ω. The bisector of $\angle BAC$ meets ω at E $(\neq A)$, the bisector of $\angle ABD$ meets ω at F $(\neq B)$, AE intersects BF at P and CF intersects DE at Q. Suppose $EF = 20$, $PQ = 11$. Find the area of the quadrilateral $PEQF$.

8. Let $f(x) = \sqrt{x^2 + 1} + \sqrt{(4 - x)^2 + 4}$. Find the minimum value of $f(x)$.

9. It is known that $a \geq 0$ satisfies $\sqrt{4 + \sqrt{4 + \sqrt{4 + \sqrt{4 + a}}}} = a$. find the value of $(2a - 1)^2$.

10. A rectangle with sides parallel to the horizontal and vertical axes is inscribed in the region bounded by the graph of $y = 60 - x^2$ and the x-axis. If the area of the largest such rectangle has area $k\sqrt{5}$, find the value of k.

11. Let x be a real number satisfying the equation $x^{x^5} = 100$. Find the value of $\lfloor x^5 \rfloor$.

12. Let a, b, c, d, e be distinct integers with $a + b + c + d + e = 9$. If m is an integer such that

$$(m-a)(m-b)(m-c)(m-d)(m-e) = 2009,$$

determine the value of m.

13. Let $\{x\}$ be the fractional part of the number x, i.e., $\{x\} = x - \lfloor x \rfloor$. If $S = \int_0^9 \{x\}^2 dx$, find $\lfloor S \rfloor$.

14. The solution of the inequality $|(x+1)(x-6)| > |(x+4)(x-2)|$ can be expressed as $x < a$ or $b < x < c$. If $S = |a| + |b| + |c|$, find $\lfloor 14S \rfloor$.

15. Given that $x, y > 0$ and $x\sqrt{2-y^2} + y\sqrt{2-x^2} = 2$, find the value of $x^2 + y^2$.

16. A convex polygon has n sides such that no three diagonals are concurrent. It is known that all its diagonals divide the polygon into 2500 regions. Determine n.

17. Find the number of integers n between -2029 and 2029 inclusive such that $(n+2)^2 + n^2$ is divisible by 2029.

18. Let f be a function such that for any real number x,we have$f(x) + 2f(2-x) = x + x^2$. Find the value of $f(1) + f(2) + f(3) + \cdots + f(34)$.

19. Find the largest positive prime integer p such that p divides

$$S(p) = 1^{p-2} + 2^{p-2} + 3^{p-2} + 4^{p-2} + 5^{p-2} + 6^{p-2} + 7^{p-2} + 8^{p-2}.$$

20. Let f be a function such that $f(x) + f\left(\frac{1}{1-x}\right) = 1 + \frac{1}{x}$ for all $x \notin \{0, 1\}$. Find the value of $\lfloor 180 \cdot f(10) \rfloor$.

21. Let C be the circle with equation $(x-a)^2 + (y-b)^2 = r^2$, where at least one of the a and b are irrational numbers. Find the maximum possible number of points (p, q) on C where both p and q are rational numbers.

22. On the plane there are 2024 points coloured either red or blue such that each red point is the centre of a circle passing through 3 blue points. Determine the least number of blue points.

23. It is given that the positive real numbers $x_1, \ldots, x_{2026}$ satisfy $\frac{x_1^2}{x_1^2+1} + \cdots + \frac{x_{2026}^2}{x_{2026}^2+1} = 2025$. Find the maximum value of $\frac{x_1}{x_1^2+1} + \cdots + \frac{x_{2026}}{x_{2026}^2+1}$.

24. Let n denote the numbers of ways of arranging all the letters of the word MATHEMATICS in one row such that

(1) both M's precede both T's; and

(2) neither the two M's nor the two T's are next to each other.

Determine the value of $\frac{n}{6!}$.

25. The incircle of the triangle ABC centered at I touches the sides BC, CA, AB at D, E, F, respectively. Let D' be the intersection of the extension of ID with the circle through B, I, C; E' the intersection of the extension of IE with the circle through A, I, C; and F' the intersection of the extension of IF with the circle through A, I, B. Suppose $AB = 52, BC = 56, CA = 60$. Find $DD' + EE' + FF'$.