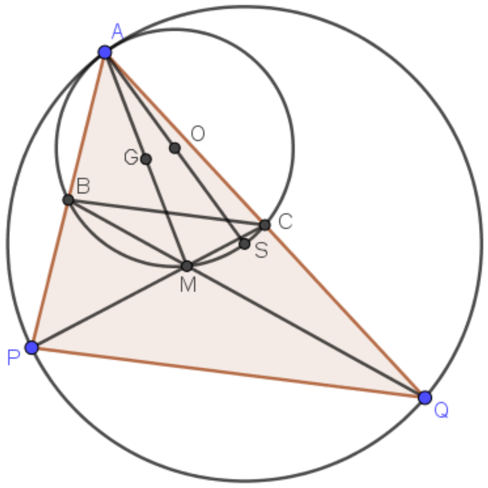
SMO Open 2023 Rd.2

1. In a scalene triangle ABC with centroid G and circumcircle ω centred at O, the extension of AG meets ω at M, lines AB and CM intersect at P; and lines AC and BM intersect at Q. Suppose the circumcentre S of the triangle APQ lies on ω and A, O, S are collinear. Prove that $\angle AGO = 90°$.



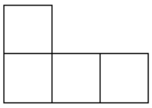
Solution:

Consider the homothety h centred at A which sends O to S and ω to the circumcentre of ΔAPQ. Since $AS = 2AO$, h has scale factor 2, so B, C are midpoints of AP, AQ respectively.

This implies that M is the centroid of ΔAPQ. Thus, $h(G) = M$, which implies $GO||MS$. Thus $\angle AGO = \angle AMS = 90°$.

Remark: A homothety is just a scaling with respect to a particular point (A, in this case). If one is unfamiliar with the concept, the statements can easily be replaced by a series of claims that use similar triangles (or the midpoint theorem, in this case.)

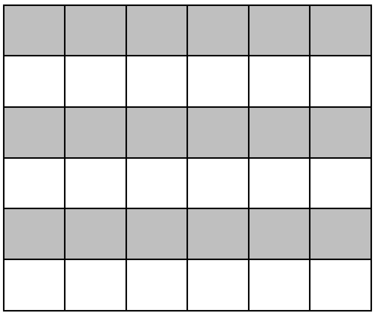
2. A grid of cells is tiled with dominoes such that every cell is covered by exactly one domino. A subset S of dominoes is chosen. Is it true that at least one of the following 2 statements is false?
   (1) There are 2022 more horizontal dominoes than vertical dominoes in S.
   (2) The cells covered by the dominoes in S can be tiled completely and exactly by the tetrominoes shown below (including rotations & reflections):



Solution:

Yes, at least one of the statements is false.

Suppose otherwise that both statements (1) and (2) are true, and let S contain n vertical and $n + 2022$ horizontal dominoes. Then there would be $n + 1011$ L-shapes.



Shade the grid such that the rows are alternately shaded and unshaded as shown above. Note that:

- A horizontal domino has 2 shaded or 2 unshaded squares.
- A vertical domino has 1 shaded and 1 unshaded square.
- An L-shape has 3 shaded and 1 unshaded square, or vice versa.

This means that the number of shaded squares is congruent to n (mod 2) since there are n vertical dominoes, but also $n + 1011$ (mod 2) since there are $n + 1011$ L-shapes, contradiction.

Remark: Colouring-based ideas would be the first things to consider when handling a 'tiling' problem, especially if we want to show something is impossible.

3. Let $n \geq 2$ be a positive integer. For any integer a, let $Q_a(x)$ denote the polynomial $x^n + ax$. Let p be a prime number and S_a be the set

$$S_a = \{b|0 \leq b \leq p-1, \exists c \in \mathbb{Z}, Q_a(c) \equiv b \pmod{p}\}.$$

Show that the expression $\frac{1}{p-1}\sum_{a=1}^{p-1}|S_a|$ is an integer.

Solution:

We define (a, b) to be an *achievable* pair if $x^n + ax \equiv b$ (mod n) has a solution, $1 \leq a \leq p-1$ and $1 \leq b \leq p-1$. (All subsequent congruences and values are taken modulo n.)

Then $\sum_{a=1}^{p-1}|S_a|$ is simply the number of achievable pairs, plus the $p-1$ pairs $(1,0), (2,0), \ldots, (p-1,0)$ which are attained when $x = 0$. So it remains to show that the number of achievable pairs is a multiple of $p-1$. Moreover, we can assume WLOG that $x \not\equiv 0$.

Suppose that (a_0, b_0) is an achievable pair, and $x^n + a_0 x \equiv b_0$.

Then for any $1 \leq k \leq p-1$,

$$k^n x^n + k^n a_0 x \equiv k^n b_0$$

$$\Leftrightarrow (kx)^n + (a_0 k^{n-1})(kx) \equiv k^n b_0$$

This implies that $(a_0 k^{n-1}, b_0 k^n)$ is also an achievable pair.

In general, for each set of the form $S_{a,b} = \{(ak^{n-1}, bk^n): 1 \leq k \leq p-1\}$ (call these type S), the pairs are either all achievable, or all not achievable.

Claim: The set of all pairs $T = \{(a, b): 1 \leq a, b \leq p-1\}$ is a disjoint union of type S sets. Moreover, each type S set has size $p-1$. (This will imply that the number of achievable pairs is a multiple of $p-1$ as desired.)

Proof: Clearly, each pair (a, b) lies in $S_{a,b}$, and $S_{a,b} \subset T$ for any a, b. We are left to show two claims:

a) Each set $S_{a,b}$ has size $p-1$.

Suppose that $(ak^{n-1}, bk^n) = (ak_0^{n-1}, bk_0^n)$ for some k, k_0. Then $\left(\frac{k}{k_0}\right)^{n-1} \equiv \left(\frac{k}{k_0}\right)^n \equiv 1$, so $\frac{k}{k_0} \equiv 1$, and $k = k_0$. Hence, each set has $p-1$ distinct elements.

b) The sets $S_{a,b}$ and $S_{a',b'}$ are either equal or disjoint.

Suppose $S_{a,b} \cap S_{a',b'} \neq \emptyset$. Then $(ak^{n-1}, bk^n) = (a'k_0^{n-1}, b'k_0^n)$ for some k, k_0.

Then $(a, b) = \left(a' \left(\frac{k_0}{k}\right)^{n-1}, b' \left(\frac{k_0}{k}\right)^{n}\right)$.

Let $(ak_1^{n-1}, bk_1^n) \in S_{a,b}$. Then $(ak_1^{n-1}, bk_1^n) = \left(a' \left(\frac{k_0 k_1}{k}\right)^{n-1}, b' \left(\frac{k_0 k_1}{k}\right)^{n}\right) \in S_{a',b'}$. So $S_{a,b} \subseteq S_{a',b'}$. Since both sets have size $p - 1$, $S_{a,b} = S_{a',b'}$.

Hence, any sets are either disjoint or equal, showing that they are disjoint sets of size $p - 1$ whose union is T.

Remark: This is also 2022 Open Q5, almost verbatim. It is odd that an SMO Open problem would repeat literally 1 year later. Although we shouldn't expect this to become a pattern, it would be a good idea to familiarize ourselves with recent SMO solutions for future contests, just in case.

4. Find all functions $f: \mathbb{Z} \to \mathbb{Z}$ such that
$$f(x+y)\left(\left(f(x)-f(y)\right)^2+f(xy)\right)=f(x^3)+f(y^3).$$

Solution:

Let $P(x,y)$ be the assertion that $f(x+y)\left(\left(f(x)-f(y)\right)^2+f(xy)\right)=f(x^3)+f(y^3)$.

$P(0,0)$: $f(0)^2=2f(0) \Rightarrow f(0)=0$ or $f(0)=2$.

$P(1,1)$: $f(2)f(1)=2f(1) \Rightarrow f(1)=0$ or $f(2)=2$.

We now split into cases.

Case 1: $f(0)=0$ and $f(1)=0$.

$P(x,0)$: $f(x)^3=f(x^3)$

$P(x,1)$: $f(x+1)\cdot\left(f(x)^2+f(x)\right)=f(x^3)=f(x)^3$.

Suppose that $f(a)\neq 0$ for some a. Then letting $x=a$ and dividing by $f(a)$,
$$f(a+1)\cdot[f(a)+1]=f(a)^2.$$
Clearly, $f(a)\neq -1$. So $f(a+1)=\frac{f(a)^2}{f(a)+1}=f(a)-1+\frac{1}{f(a)+1}$.

This is only an integer if $f(a)=-2$. Thus, the only non-zero value in the range of f should be -2. But $f(a^3)=f(a)^3=-8$, contradiction.

Hence, $\boldsymbol{f(x)=0}$ **for all** $\boldsymbol{x}$. [1st Solution]

Case 2: $f(0)=0$, $f(2)=2$ (and $f(1)\neq 0$).

As before, $P(x,0)$: $f(x)^3=f(x^3)$. In particular, this means $f(1)=-1$ or 1, and $f(-1)=-1,0$ or 1.

$P(2,-1)$: $f(1)\cdot\left(\left(f(2)-f(-1)\right)^2+f(-2)\right)=f(8)+f(-1)$
$$\Rightarrow f(1)\cdot\left(\left(2-f(-1)\right)^2+f(-2)\right)=8+f(-1)\ (*)$$
$P(-1,-1)$: $f(-2)\cdot f(1)=2f(-1)$.

If $f(-1)=0$, then since $f(1)\neq 0$, $f(-2)=0$. But substituting into $(*)$ gives $f(1)=2$, contradiction. This implies $f(1)=\pm1, f(-1)=\pm1$ and $f(-2)=\pm2$. Checking, the only combination which works is $f(1)=1, f(-1)=-1$ and $f(-2)=-2$.

Now $P(x,1)$ gives $f(x+1)\cdot[f(x)^2-f(x)+1]=f(x)^3+1$, which implies $f(x+1)=f(x)+1$ for all x. Clearly, this gives $\boldsymbol{f(x)=x}$ **for all** $\boldsymbol{x}$ [2nd solution].

Case 3: $f(0)=2$, $f(1)=0$

Then $P(1,0)$ gives $0=2$, contradiction.

Case 4: $f(0) = 2, f(2) = 2$ (and $f(1) \neq 0$).

$P(x, 0)$: $f(x)(f(x)^2 - 4f(x) + 6) = f(x^3) + 2$.
For $x = 1$ or -1, $x^3 = x$, so $f(x)(f(x)^2 - 4f(x) + 6) = f(x) + 2$.
Factorising, $f(x)^3 - 4f(x)^2 + 5f(x) - 2 = 0 \Rightarrow (f(x) - 2)(f(x) - 1)^2 = 0 \Rightarrow f(x) = 1$ or 2. Thus, $f(1)$ and $f(-1) = 1$ or 2.
$P(1, -1)$ gives $f(0) \cdot ((f(1) - f(-1)^2 + f(-1)) = f(1) + f(-1)$. Since $f(0) = 2$, $f(1) + f(-1)$ are even, i.e. $f(1) = f(-1) = 1$ or $f(1) = f(-1) = 2$.

Case 4a: $f(1) = f(-1) = 2$.
We show by induction on m that $f(x) = 2$ for $x \in [-m, m]$.
It is true for $m = 1$; suppose it is true for $m = k - 1$.
Note that from $P(x, 0)$, $f(x) = 2 \Rightarrow f(x^3) = 2$ as well.
Then $P(k - 1, 1)$: $f(k)\left(\left(f(k-1) - f(1)\right)^2 + f(k-1)\right) = f((k-1)^3) + f(1)$

$$\Rightarrow f(k) \cdot 2 = 2 + 2$$
$$\Rightarrow f(k) = 2.$$

Likewise, $P(-(k-1), -1)$ gives $f(-k) = 2$.
This completes the induction, giving $\boldsymbol{f(x) = 2}$ **for all** $\boldsymbol{x}$ [3rd solution].

Case 4b: $f(1) = f(-1) = 1$.
Similarly, we may show by induction on m that $f(x) = 1$ for odd x and $f(x) = 2$ for even x where $x \in [-m, m]$.

So $\boldsymbol{f(x) = \begin{cases} 1, & x \text{ odd} \\ 2, & x \text{ even} \end{cases}}$ also works [4th solution].

Remark: This is a very tedious question – but none of the steps are particularly clever. It is just <u>very</u> long. We know that a much faster solution is unlikely to exist, since there are four different $f(x)$ that all work!

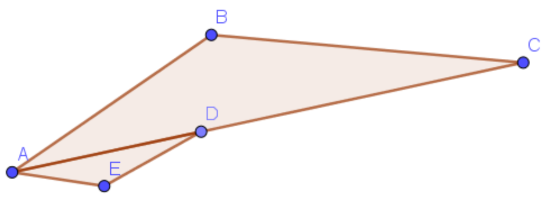
5. Determine all real numbers x between 0 and 180 such that it is possible to partition an equilateral triangle into finitely many triangles, each of which has an angle of $x°$.

Solutions:

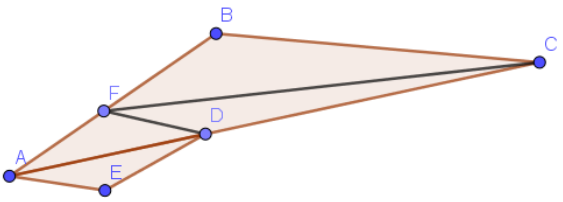
We claim that all $0 < x \leq 120$ work.

<u>Proof of Impossibility:</u>

Call a triangle with a $> 120°$ angle *large*. We simply show that we cannot partition the equilateral triangle into large triangles. For convenience, we claim that WLOG, we may assume that no triangles 'partially' share an edge. Suppose we have such a situation (e.g. AD lying on AC).



Then we further split ΔBCA into large triangles by letting F be on AB so that $FD||BC$:



Now let there be m triangle vertices on the sides of the original equilateral triangle (excluding the 3 original vertices), and n vertices in the interior which do not lie on the side of any triangle.

Then there are

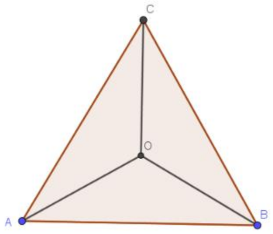
- No $x°$ angles on each of the 3 original vertices
- ≤ 1 $x°$ angle on each of the m "side" vertices
- ≤ 2 $x°$ angles on each of the n "interior" vertices.

So we have at most $m + 2n$ angles $> 120°$.

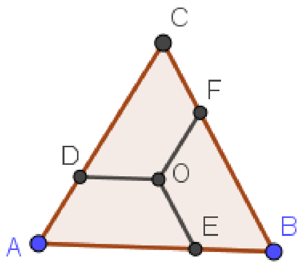
However, the sum of angles is $3 \times 60° + m \times 180° + n \times 360°$, giving $1 + m + 2n$ triangles, so some triangle does not have a $> 120°$ angle.

Constructions:

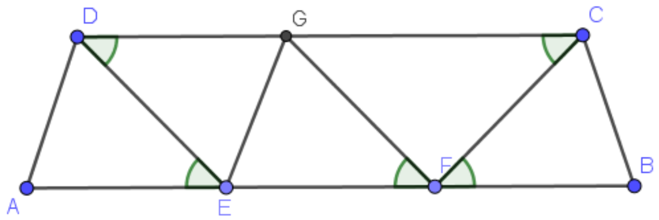
$x = 120°$ is simple; we just divide into three $120 - 30 - 30$ triangles.



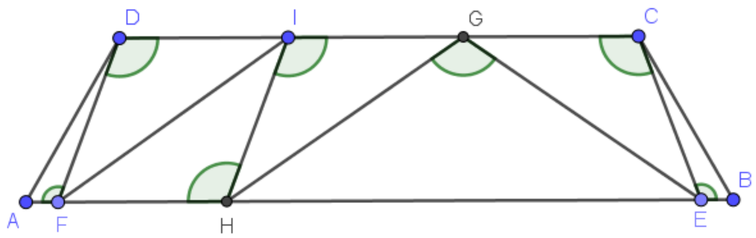
Otherwise, we split the equilateral triangle into 3 congruent trapeziums as shown, where O is the centroid of the equilateral triangle ABC.



It suffices to consider a particular trapezium, say, $ADOE$. Note that by drawing parallel lines to OD, we can divide it into smaller trapeziums with angles $60°, 60°, 120°, 120°$, and whose bases are at least k times of the height for arbitrarily large k. We show how to divide such a trapezium into triangles with angle $x°$.



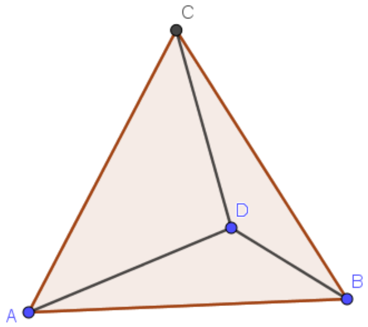
For $x < 90$, we construct on trapezium $ABCD$ (where $AB||CD$ and $AB > CD$) by letting E, F be on AB such that $\angle AED = \angle BFC = x°$, then letting G be on CD such that $DE||GF$, which ensures $\angle GDE = \angle GFE = \angle GCF = x°$.



For $x > 90$, we let E, F be on AB again so that $\angle AFD = \angle CEB = x°$. Then we let G be on CD so that $CE = CG$, reflect C over G onto I, and let H be on AB so that $HI||FD$. This ensures that all triangles have a $x°$ angle again.

Remarks: This is a nice, but difficult question. It is tempting to guess that something like $x = \frac{60}{n}, \frac{90}{n}, \frac{120}{n}$ *or* $\frac{360}{n}$ $(n \in \mathbb{Z}^+)$ *would be the answer.*

To motivate the answer ***not*** *being one of these, we may consider a simple configuration:*



It is very possible that for suitable D*,* $\angle BCD, \angle ADB$ *and* $\angle CAD$ *are multiples of, say,* $7°$ *or* $1.7°$ *- and more importantly, it seems impossible to show that this* ***never*** *happens! Since there's no reason to think that any particular angle is 'special enough', which suggests that an entire range of values should be achievable.*