# Singapore Mathematical Society
# Singapore Mathematical Olympiad (SMO) 2023
## (Open Section, Round 1)

**Wednesday, 31 May 2023** **0930-1200 hrs**

**Instructions to contestants**

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

**PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO**

In this paper, let $\mathbb{R}$ denote the set of all real numbers, and $\lfloor x \rfloor$ denote the greatest integer not exceeding $x$. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$.

1. The graph $C$ with equation $y = \dfrac{ax^2 + bx + c}{x + 2}$ has an oblique asymptote with equation $y = 4x - 6$ and one of the stationary points at $x = -4$. Find the value of $a + b + c$.

2. If $x = \dfrac{1}{1} + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \dfrac{1}{1+2+3+4} + \cdots + \dfrac{1}{1+2+3+\cdots+100}$, find the value of $\lfloor 1010x \rfloor$.

3. The set of all possible values of $x$ for which the sum of the infinite series
$$1 + \frac{1}{6}(x^2 - 5x) + \frac{1}{6^2}(x^2 - 5x)^2 + \frac{1}{6^3}(x^2 - 5x)^3 + \cdots$$
exists can be expressed as $(a, b) \bigcup (c, d)$, where $a < b < c < d$. Find $d - a$.

4. Find the value of $\lfloor y \rfloor$, where $y = \displaystyle\sum_{k=0}^{\infty} (2k + 1)(0.5)^{2k}$.

   (Hint: Consider the series expansion of $(1 - x)^{-2}$.)

5. The solution of the inequality $|x - 1| + |x + 1| < ax + b$ is $-1 < x < 2$. Find the value of $\lfloor a + b \rfloor$.

6. The equation $x^4 - 4x^2 + qx - r = 0$ has three equal roots. Find the value of $\left\lfloor \dfrac{3q^2}{r^2} \right\rfloor$.

7 The parabolas $y = x^2 - 16x + 50$ and $x = y^2$ intersect at 4 distinct points which lie on a circle centred at $(a, b)$. Find $|a - b|$.

8. In the 3-dimensional Euclidean space with origin $O$ and three mutually perpendicular $x$-, $y$- and $z$-axes, two planes $x + y + 3z = 4$ and $2x - z = 6$ intersect at the line $\mathbf{r} \times \begin{pmatrix} -1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ c \\ d \end{pmatrix}$. Find the value of $|a + b + c + d|$.

9. Let $x, y, z$ be real numbers with $3x + 4y + 5z = 100$. Find the minimum value of $x^2 + y^2 + z^2$.

10. Find the area of the region represented by the equation $\lfloor x \rfloor + \lfloor y \rfloor = 1$ in the region $0 \leq x < 2$.

    (Note: If you think that there is no area defined by the graph, enter "0"; if you think that the area is infinite, enter "9999".)

11. Let $ABC$ be a triangle satisfying the following conditions that $\angle A + \angle C = 2\angle B$, and $\frac{1}{\cos A} + \frac{1}{\cos C} = \frac{-\sqrt{2}}{\cos B}$. Determine the value of $\dfrac{2022\cos(\frac{A-C}{2})}{\sqrt{2}}$.

12. Find $x$ which satisfies the following equation

$$\frac{x-2019}{1} + \frac{x-2018}{2} + \frac{x-2017}{3} + \cdots + \frac{x+2}{2022} + \frac{x+3}{2023} = 2023.$$

13. Assume that $x$ is a positive number such that $x - \frac{1}{x} = 3$ and

$$\frac{x^{10}+x^8+x^2+1}{x^{10}+x^6+x^4+1} = \frac{m}{n},$$

where $m$ and $n$ are positive integers without common factors larger than 1. Determine the value of $m+n$.

14. Consider the set of all possible pairs $(x, y)$ of real numbers that satisfy $(x-4)^2 + (y-3)^2 = 9$. If $S$ is the largest possible value of $\frac{y}{x}$, find the value of $\lfloor 7S \rfloor$.

15. Let $x, y$ be positive integers with $16x^2 + y^2 + 7xy \leq 2023$. Find the maximum value of $4x + y$.

16. Let $x$ be the largest real number such that

$$\sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} = x.$$

Determine the value of $(2x-1)^4$.

17. Two positive integers $m$ and $n$ differ by 10 and the digits in the decimal representation of $mn$ are all equal to 9. Determine $m+n$.

18. Let $\{a_n\}$ be a sequence of positive numbers, and let $S_n = a_1 + a_2 + \cdots + a_n$. For any positive integer $n$, let $b_n = \frac{1}{2}\left(\frac{a_{n+1}}{a_n} + \frac{a_n}{a_{n+1}}\right)$. Given that $\frac{a_n+2}{2} = \sqrt{2S_n}$ holds for all positive integers $n$, determine the limit $\lim_{n\to+\infty}(b_1 + b_2 + \cdots + b_n - n)$.

19. Let $ABC$ be a triangle with $AB = c, AC = b$ and $BC = a$, and satisfies the conditions $\tan C = \frac{\sin A + \sin B}{\cos A + \cos B}$, $\sin(B - A) = \cos C$ and that area of triangle $ABC = 3 + \sqrt{3}$. Determine the value of $a^2 + c^2$.

20. Let $g: \mathbb{R} \to \mathbb{R}$, $g(0) = 4$ and that

$$g(xy+1) = g(x)g(y) - g(y) - x + 2023.$$

Find the value of $g(2023)$.

21. In the triangle $ABC$, $D$ is the midpoint of $AC$, $E$ is the midpoint of $BD$, and the lines $BA$ and $CE$ are tangent to the circumcircle of the triangle $ADE$ at $A$ and $E$ respectively. Suppose the circumradius of the triangle $AED$ is $(\frac{64}{7})^{\frac{1}{4}}$. Find the area of the triangle $ABC$.

22. $ABCD$ is a parallelogram such that $\angle ABC < 90°$ and $\sin \angle ABC = \frac{4}{5}$. The point $K$ is on the extension of $BC$ such that $DC = DK$; the point $L$ is on the extension of $DC$ such that $BC = BL$. The bisector of $\angle CDK$ intersects the bisector of $\angle LBC$ at $Q$. Suppose the circumradius of the triangle $ABD$ is 25. Find the length of $KL$.

23. A group of 200 monkeys is given the task of picking up all 3000 peanuts on the ground. Determine the maximum number $k$ such that there must be $k$ monkeys picking up the same number of peanuts.

[It is possible that some lazy monkeys may not pick up any peanuts at all.]

24. A chain of $n$ identical circles $C_1, C_2, \ldots, C_n$ of equal radii and centres on the $x$-axis lie inside the ellipse $E$: $\frac{x^2}{2023} + \frac{y^2}{333} = 1$ such that $C_1$ is tangent to $E$ internally at $(-\sqrt{2023}, 0)$, $C_n$ is tangent to $E$ internally at $(\sqrt{2023}, 0)$, and $C_i$ is tangent to $C_{i+1}$ externally for $i = 1, \ldots, n-1$. Determine the smallest possible value of $n$.

25. Let $p > 2023$ be a prime. Determine the number of positive integers $n$ such that

$$(n-p)^2 + 2023(2023 - 2n - 2p)$$

is a perfect square.