Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2021
(Open Section, Round 1)

Thursday, 3 June 2021 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

In this paper, let $\mathbb{R}$ denote the set of all real numbers, and $\lfloor x \rfloor$ denote the greatest integer not exceeding x and let $\lceil x \rceil$ denote the smallest integer not less than x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$; $\lceil 5 \rceil = 5$, $\lceil 2.8 \rceil = 3$, and $\lceil -2.3 \rceil = -2$.

1. It is given that $\frac{\pi}{2} < \beta < \alpha < \frac{3\pi}{4}$, $\cos(\alpha - \beta) = \frac{12}{13}$ and $\sin(\alpha + \beta) = -\frac{3}{5}$. Find $\lfloor |2021 \sin(2\alpha)| \rfloor$.

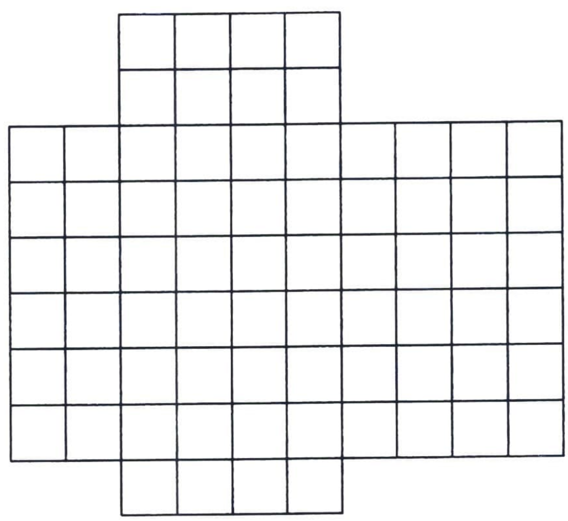
2. Find the number of solutions of the equation $|x-3| + |x-5| = 2$.

   (Note: If you think that there are infinitely many solutions, enter your answer as "99999".)

3. Evaluate $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \cdots + 10 \times 11 \times 12$.

4. It is given that the solution of the inequality $\sqrt{81 - x^4} \leq kx + 1$ is $a \leq x \leq b$ with $b - a = 2$, where $k > 0$. Determine $\lfloor k \rfloor$.

5. The figure below shows a cross that is cut out from a 10×9 rectangular board.



   Find the total number of rectangles in the above figure.

   (Note: A square is a rectangle.)

6. Consider all the polynomials $P(x, y)$ in two variables such that $P(0,0) = 2020$ and for all x and y, $P(x, y) = P(x+y, y-x)$. Find the largest possible value of $P(1,1)$.

7. In the three dimensional Cartesian space with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denoting the unit vectors along three perpendicular directions in a clockwise manner, the line l with equation given by $\mathbf{r} \times (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 5\mathbf{i} - 13\mathbf{j} + 7\mathbf{k}$ intersects the plane Π with equation $x + y + z = 16$ at the point (a, b, c). Find the value of $a + b + c$.

8. Find the minimum value of $(x+7)^2 + (y+2)^2$ subject to the constraint $(x-5)^2 + (y-7)^2 = 4$.

9. Find the largest possible value of $\alpha^4+\beta^4+\gamma^4$ among all possible sets of numbers (α,β,γ) that satisfy the equations

$$\begin{aligned}\alpha+\beta+\gamma &= 2\\ \alpha^2+\beta^2+\gamma^2 &= 14\\ \alpha^3+\beta^3+\gamma^3 &= 20.\end{aligned}$$

10. If p is the product of all the non-zero real roots of the equation

$$\sqrt[9]{x^7+30x^5}=\sqrt[7]{x^9-30x^5},$$

find $\lfloor |p| \rfloor$.

11. Let S be the sum of a convergent geometric series with first term 1. If the third term of the series is the arithmetic mean of the first two terms, find $\lfloor 3S+4 \rfloor$.

12. Given that $\sin\alpha+\sin\beta=\frac{1}{10}$, and $\cos\alpha+\cos\beta=\frac{1}{9}$, find $\lfloor \tan^2(\alpha+\beta) \rfloor$.

13. Determine the number of positive integers that are divisible by 2021 and has exactly 2021 divisors (including 1 and itself).

14. Let $S=\sum_{k=0}^{25}\binom{100}{4k}-2^{98}$. Find $\left\lfloor \left| \frac{S}{2^{48}} \right| \right\rfloor$.

15. Assume that ABC is an acute triangle with $\sin(A+B)=\frac{3}{5}$ and $\sin(A-B)=\frac{1}{5}$. If $AB=2022(\sqrt{6}-2)$, determine $\lfloor h \rfloor$, where h is the height of the triangle from C on AB.

16. Let $a_1,a_2,\cdots$ be a sequence with $a_1=1$ and $a_{n+1}=\frac{n+2}{n}S_n$ for all $n=1,2,\cdots$, where $S_n=a_1+a_2+\cdots+a_n$. Determine the minimum integer n such that $a_n\geq 2021$.

17. Each card of a stack of 101 cards has one side colored red and the other colored blue. Initially all cards have the red side facing up and stacked together in a deck. On each turn, Ah Meng takes 8 cards on the top, flip them over, and place them to the bottom deck. Determine the minimum number of turns required so that all the cards have the red sides facing up again.

18. Let ABC be a triangle with $AB=10$ and $\frac{\cos A}{\cos B}=\frac{AC}{BC}=\frac{4}{3}$. Let P be a point on the inscribed circle of triangle ABC. Find the largest possible value of $PA^2+PB^2+PC^2$.

19. A basket contains 19 apples labeled by the numbers 2,3,...,20, and 19 bananas labeled by the numbers 2,3,...,20. Ah Beng picks m apples and n bananas from the basket. However he needs to ensure that for any apple labeled a and any banana labeled b that he picks, a and b are relatively prime. Determine the largest possible value of mn.

20. Let $p(x) = ax^2 - bx + c$ be a polynomial where a, b, c are positive integers and $p(x)$ has two distinct roots in $(0, 1)$. Determine the least possible value of abc.

21. In the triangle ABC, $\angle A > 90°$, the incircle touches the side BC and AC at A_1 and B_1 respectively. The line A_1B_1 meets the extension of BA at X such that $\angle CXB = 90°$. Suppose $BC^2 = AB^2 + BC \cdot AC$. Find the size of $\angle A$ in degrees.

22. Find the number of positive integers n such that $7n - 16$ divides $n \cdot 13^{2019}$.

23. In the acute triangle ABC, P is a point on AB, Q is a point on AC such that $BP + CQ = PQ$. The bisector of $\angle A$ meets the circumcircle of the triangle ABC at the point R distinct from A. Suppose $\angle PRQ = 52.5°$. Find the size of $\angle BAC$ in degrees.

24. Let $S = \displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx$. Determine the value of $\lfloor S^2 \rfloor$.

25. Let p, q, r be positive numbers with $p - r = 4q$, and $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ be two sequences defined by $a_1 = p, b_1 = q$ and for $n \geq 2$,

$$a_n = pa_{n-1}, b_n = qa_{n-1} + rb_{n-1}.$$

Find the value of $\displaystyle\lim_{n\to\infty} \frac{\sqrt{a_n^2 + (3b_n)^2}}{b_n}$.