Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2018
(Open Section, Round One)

Thursday, 31 May 2018 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

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In this paper, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$.

1. Find the area of the region R on the xy-plane consisting of the points (x, y) satisfying the equation $\lfloor x \rfloor + \lfloor y \rfloor = 10$, where $0 \leq x \leq 3$.

   (Note: If you think that the region R does not have any area, enter your answer as "0"; if you think that the region R is unbounded, enter your answer as "99999".)

2. Find the sum

$$1 + 1 + 2 + 1 + 2 + 3 + 1 + 2 + 3 + 4 + \cdots + 1 + 2 + 3 + \cdots + 20.$$

3. Let a and b be the largest and smallest values of x that satisfy the equation

$$|x - 1| + |6 - 2x| = |5 - x|.$$

   Find $a - b$.

4. Let $S_n = \sum_{k=1}^{n} \frac{k}{(k+1)!}$. Find the value of $2019! \times (1 - S_{2018})$.

5. A rectangular table has two chairs on each of the longer sides and one chair on each of the shorter sides. In how many ways can six people be seated?

   (Note: Any two arrangements are the same up to 'rotation' of the rectangular table.)

6. Find the smallest positive integer such that the sum of the fifth power of its digits is not divisible by the sum of its digits.

7. Three of the four integers between 100 and 1000 which are equal to the sum of the cubes of their digits are 153, 371, and 407 (For example, $1^3 + 5^3 + 3^3 = 153$.). Determine the fourth integer.

8. Find the minimum value of the function $f(x) = \frac{x^2 + x + 2018}{x - 2017}$ for $x > 2017$.

9. Let $p(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are distinct non-zero integers. Suppose $p(a) = a^3$ and $p(b) = b^3$. Find $p(13)$.

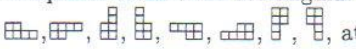
10. Find the smallest integer r such that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{2018}} \leq r\sqrt{2018}.$$

11. Find the shortest distance (rounded off to the nearest whole number if necessary) from the point $(22, 21)$ to the graph with equation $x^3 + 1 = y(3x - y^2)$.

12. Given that $S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n+k}$, find the value of e^S, where e is the base of the natural logarithm.

13. Let $\triangle ABC$ be a triangle with $a = BC, b = AC$ and $c = AB$. Assume that $a + c = 2b$, $\angle A - \angle C = \frac{\pi}{3}$ and $\sin B = \frac{\sqrt{m}}{n}$ for some positive integers m, n. What is the least possible value of $m + n$?

14. Ah Meng is going to shade some squares of an 6×11 rectangular board in such a way that in every L-shape of the forms , at least half of the squares in it are shaded. Determine the smallest number of squares in the board that Ah Meng must shade in order to fulfill this condition.

15. In the triangle ABC, $AB = 7, BC = 10$ and $CA = \sqrt{73}$, M is the midpoint of AC, and P is the point on BC such that AP intersects BM at Q and $BP = BQ$. Find the length of AP.

16. A triangle $A'B'C'$ is formed with sides whose lengths are the lengths of the medians of a triangle ABC. Suppose the product of the lengths of the three sides of ABC is 640. Find the product of the lengths of the medians of the triangle $A'B'C'$.

17. Let $f_0(x) = \frac{x}{3x+2}$ and for any integer $n \geq 1$, $f_n(x) = f_0(f_{n-1}(x))$. Assume that $f_{2018}(x) = \frac{x}{Ax+B}$. What is the value of $3B - A$?

18. How many 3-element subsets $\{a, b, c\}$ of $\{1, 2, \cdots, 100\}$ have the property that $a + b + c$ is a multiple of 6? For example, $\{1, 2, 3\}$ and $\{2, 6, 10\}$ are examples of such sets.

(Note: $\{a, b, c\}$, $\{a, c, b\}$ and $\{b, a, c\}$ are considered as the same set.)

19. Assume that $a_1 < 2$, and for any integer $n \geq 2$, $a_n = 1 + a_{n-1}(a_{n-1} - 1)$. If

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_m} = 1$$

for some integer m, what is the maximum value of $16a_1 - a_{m+1}$?

20. Let a, b, c be real numbers such that $a + bc = b + ca = c + ab = 870$. Determine the maximum value of $ab + bc + ca$.

21. Determine the largest value of the expression

$$2^{k_1} + 2^{k_2} + \cdots + 2^{k_{498}},$$

where for each $i = 1, \ldots, 498$, k_i is an integer, $1 \leq k_i \leq 507$, and $k_1 + \cdots + k_{498} = 507$.

22. Some persons in a party shake hands with each other. The following information is known.

- Each person shakes hands with exactly 20 persons.
- For each pair of persons who shake hands with each other, there is exactly 1 other person who shake hands with both of them.
- For each pair of persons who do not shake hands with each other, there are exactly 6 other persons who shake hands with both of them.

Determine the number of persons in the party.

23. In a rectangle $ABCD$, E is a point on AD and F is a point on CD such that the line through the midpoint of EF and the centre of the rectangle is perpendicular to AC. Given $AB = 100, BC = 60$ and $AE = 40$, find the area of the triangle BEF.

24. Let a, b, c be positive numbers such that $a + b + c = 2$. If the minimum value of

$$(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2 + (c + \frac{1}{c})^2$$

is $\frac{m}{n}$, where m and n have no common factors larger than 1, find the value of $m + n$.

25. In a triangle ABC, $AB = 21, BC = 27$ and $CA = 24$, a circle ω is tangent to the sides AB and AC and is also tangent internally to the circumcircle of ABC. Let the inradius of the triangle ABC be r and the radius of ω be k. Find the value of rk.