Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2017
(Open Section, Round One)

Wednesday, 31 May 2017 0930-1200 hrs

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

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In this paper, let $\mathbb{R}$ denote the set of all real numbers. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$.

1. Let x be a real number such that $S = \dfrac{\sqrt{4-x^2}+\sqrt{x^2-4}}{|x-2|} + (5+x)^{2017}$ is an integer. Find the units digit of S.

2. Find the value of $\sqrt{17+\sqrt{288}} - \sqrt{7+\sqrt{48}} + \sqrt{17-\sqrt{288}} - \sqrt{7-\sqrt{48}}$.

3. If $T = \dfrac{1}{1} + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \dfrac{1}{1+2+3+4} + \cdots + \dfrac{1}{1+2+3+4+\cdots+2016}$, find $2017 \times T$.

4. Find the value of a such that $\displaystyle\int_0^a \lfloor x \rfloor \mathrm{d}x = 55$.

5. In the three-dimensional space with origin O, let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be three mutually perpendicular unit vectors defined in the usual anticlockwise sense. Two intersecting lines l and m have equations

$$l: \quad \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}), \quad \lambda \in \mathbb{R},$$

and

$$m: \quad \mathbf{r} = 4\mathbf{i} + 8\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}), \quad \mu \in \mathbb{R},$$

respectively. The line with equation

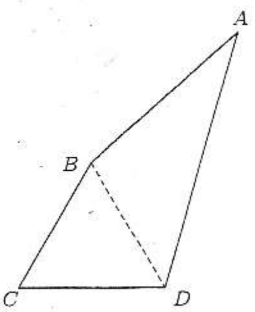
$$\mathbf{r} = \mathbf{i} + a\mathbf{j} + b\mathbf{k} + \omega(c\mathbf{i} + \mathbf{j}), \quad \omega \in \mathbb{R}$$

intersects the lines l and m and bisects the angle subtended by l and m. Find the value of $|a+b+c|$.

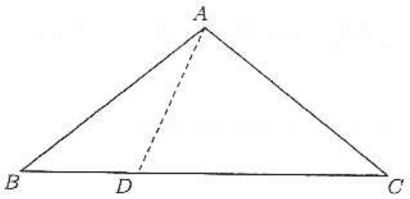
6. Let $a = \sqrt{7} - \lfloor \sqrt{7} \rfloor$. Find the value of $a^2 + 2\sqrt{7} + 2a$.

7. In how many ways can 2017 be expressed as the sum of one or more positive integers in non-decreasing order, such that the difference between the last term and the first term is at most 1?

8. In the given quadrilateral $ABCD$, $BC = CD = BD$, $AB = 6$, $AD = 8$ and $\angle BAD = 30^\circ$. Find AC.



9. Given that in triangle ABC in which $AB = AC$ and $\angle BAC = 120°$, D is the point on BC such that $BD = 10$ and $DC = 20$. Find AD.



10. In the isosceles triangle ABC, $AB = AC = 40$, $BC = 20\sqrt{3}$, and P is a point on the segment AB. The circumcircle of the triangle PBC intersects the segment AC at Q. Suppose the triangles APQ and PBC have the same circumradius. Find the length of AQ.

11. Let a, b, c and d be positive numbers such that $a + b + c + d = 8$, and that

$$\frac{a}{b+c+d} + \frac{b}{a+c+d} + \frac{c}{a+b+d} + \frac{d}{a+b+c} = \frac{3}{10}.$$

Find the value of $\frac{80}{b+c+d} + \frac{80}{a+c+d} + \frac{80}{a+b+d} + \frac{80}{a+b+c}$.

12. Let $z_n = \left(-\frac{3}{2} + \frac{\sqrt{3}}{2}i\right)^n$, where $i^2 + 1 = 0$. Find the least positive integer n such that $|z_{n+1} - z_n|^2 > 7000$.

13. Let $\triangle ABC$ be a triangle with $a = BC, b = AC$ and $c = AB$. If $b^2 = a^2 + bc$, $\angle A = 30°$ and $\angle C = x°$, find the value of x.

14. Find the largest possible integer n such that $1! + 2! + \cdots + n! = m^2$ for some integer m.

15. Determine the number of integers n with $1 \leq n \leq 2017$ such that 323 divides the expression $20^n + 16^n - 3^n - 1$.

16. There are 674 apples, 674 oranges and 674 pears to be packed into 2 boxes such that both boxes contain 3 types of fruits and the products of the number of apples, oranges and pears in both boxes are the same. Determine the number of ways that this can be done.

17. Let S be the set of all integers x satisfying $2017x^2 + (4036x - 1)y + 2019y^2 = 0$ for some integer y. Find the sum of all elements in S.

18. Let $f(x) = \sqrt{x^2 - 10x + 314} + \sqrt{x^2 + 20x + 325}$. Find the minimum value of $\lfloor f(x) \rfloor$ for real number x.

19. For any positive integer n, let a_n be the units digit of the sum $1+2+3+\cdots+n$. Thus $a_1 = 1, a_2 = 3, a_3 = 6, a_4 = 0$.

Determine a_r for $r = 2017^{2017}$.

20. Let $a_0 = 5$ and $a_{n+1}a_n = a_n^2 + 1$ for all $n \geq 0$. Determine $\lfloor a_{1000} \rfloor$.

21. How many 7-digit numbers formed by using only the digits 3 and 7 and divisible by 21 are there?

22. Each cell of a 20×20 table is painted with a colour so that the cells in each row and each column are coloured by at most 7 colours. Determine the maximum number of colours that can be used.

23. Let $a_1, a_2, a_3, a_4, \cdots$ be the sequence formed by adding the numbers in all the possible nonempty finite subsets of $\{3^0, 3^1, 3^2, 3^3, \cdots\}$ and are arranged as a monotone increasing sequence. Thus, $a_1 = 1(= 3^0), a_2 = 3(= 3^1), a_3 = 4(= 3^0 + 3^1), a_4 = 9(= 3^2), a_5 = 10(= 3^0 + 3^2), \cdots$ and so on. Determine the value of a_{217}.

24. Let (a_n) be a sequence with $a_{10} = 2\sqrt{3}$ and $a_n = \frac{8a_{n-1}}{4-a_{n-1}^2}$ for $n \geq 2$. Find the value of a_{2017}^2.

25. In a triangle ABC, $AB = AC$, $BC = 22\sqrt{3}$, $\cos A = -\frac{17}{225}$, D is a point on AC such that $AD < DC$ and P is the point on the segment BD such that $\angle APC = 90^\circ$. Given $\angle ABD = \angle BCP$, find BD.