Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2014
(Open Section, First round)

Wednesday, 4 June 2014 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

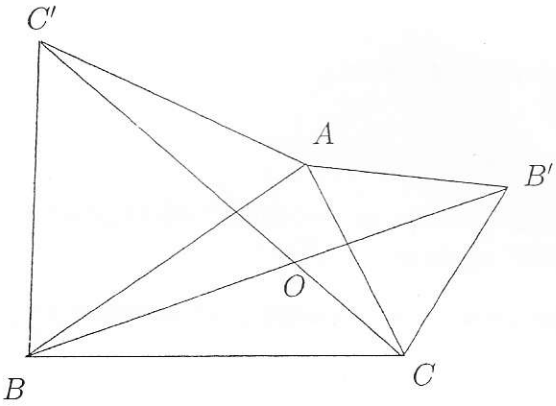
PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

In this paper, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. For examples, $\lfloor 5 \rfloor = 5$; $\lfloor 2.8 \rfloor = 2$; and $\lfloor -2.3 \rfloor = -3$.

1. Find the sum

$$1^2 \times 3 + 2^2 \times 4 + 3^2 \times 5 + 4^2 \times 6 + \cdots + 20^2 \times 22.$$

2. In the following figure, ABC is a triangle and both ABC' and $AB'C$ are equilateral triangles. Let O be the meeting point of lines CC' and BB'. Find $\angle BOC$ in degrees.



3. Consider the function $\mathrm{g}(x) = Ax^2 + Bx$, where A and B are constants. Assume that u, v are two numbers such that

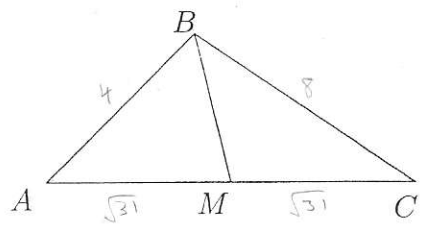
$$\mathrm{g}(u-3) = \mathrm{g}(v+3), \quad \text{and } u - v \neq 6.$$

Find the largest possible value of $A\left(\mathrm{g}(u+v)\right)^2 + B\mathrm{g}(u+v)$.

4. Let $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ be two arithmetic progressions such that $a_1 = 10$ and $b_1 = 24$, and that $a_{100} + b_{100} = 2014$. Find the sum of the first twenty terms of the sequence

$$a_1 + b_1, a_2 + b_2, a_3 + b_3, \cdots.$$

5. The figure below (not drawn to scale) shows a triangle ABC with $BC = 8$ cm, $BA = 4$ cm and $AC = 2\sqrt{31}$ cm. The point M is the midpoint of AC. Find the length of BM in centimetres.



6. If x, y and z are real numbers satisfying the equation $x^2+y^2+z^2-xy-yz-zx=27$, find the maximum value of $|y-z|$.

7. The set A is a non-empty subset of the set $\{1,2,3,4,5,6,7,8,9\}$ with the property that whenever $a \in A$, then $10-a \in A$. How many possible subsets A are there?

8. In the sequence

$$\frac{1}{4},\frac{2}{4},\frac{3}{4},\frac{1}{9},\frac{2}{9},\frac{3}{9},\cdots,\frac{8}{9},\frac{1}{16},\frac{2}{16},\cdots\frac{15}{16},\frac{1}{25},\frac{2}{25},\cdots,\frac{24}{25},\cdots$$

the N^{th} term is $\frac{2014}{2025}$. Find N.

9. A real-valued function f satisfies the equation $\mathrm{f}(x)+\mathrm{f}\left(\frac{1}{1-x}\right)=\frac{x}{x-1}$ for all real numbers $x \neq 0,1$. Find $24 \times \mathrm{f}(-3)$.

10. Given that $a=\log_2 3$, $b=-\log_4 5$ and $c=-\log_2 3+\log_4 5$, find the value of

$$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab}.$$

11. Let m be the smallest value of the function $y=\sqrt{x^2+4x+7}+\sqrt{x^2-2x+5}$. Determine $\lfloor m \rfloor$.

12. Find the number of triples (a,b,c) such that a,b,c are numbers in the set $\{1,2,3,\cdots,15\}$ satisfying the conditions $a<b-1$ and $b<c-2$.

13. The first term of an arithmetic progression is an integer and the common difference is 2. If the sum of the first n terms $(n>1)$ of the arithmetic progression is 2014, find the sum of all the possible values of n.

14. Let $a_i \in \{1,-1\}$ for all $i=1,2,3,\cdots,2014$ and

$$M=\sum_{1\leq i<j\leq 2014} a_i a_j.$$

Find the least possible positive value of M.

15. Find the largest even four-digit perfect square whose units digit exceeds the tens digit by 1 and whose thousands digit exceeds the hundreds digit by 1.

16. Let ABC be a triangle with $a = BC, b = AC$ and $c = AB$. Assume that $3a^2 + 3b^2 = 5c^2$. Find the value of

$$\frac{\cot A + \cot B}{\cot C}.$$

17. Find the number of integers k in the set $S = \{1, 2, 3, \cdots, 12014\}$ such that k is of exactly one of the following forms

$$n^2, n^3, n^5,$$

where n is an integer.

(Note: For example, $100, 1000$ are such numbers but $64 = 8^2 = 4^3$ is not.)

18. Let a, b, c be positive real numbers such that $abc = 1$. Find the least possible value of

$$\frac{2014}{a^3(b+c)} + \frac{2014}{b^3(a+c)} + \frac{2014}{c^3(a+b)}.$$

19. A sequence $a_0, a_1, a_2, a_3, \cdots$, with $a_1 = 1$, is defined such that for any positive integers m and n, where $m \geq n$, the terms of the sequence satisfy the relation

$$a_{m+n} + a_{m-n} + m - n = \frac{1}{2}(a_{2m} + a_{2n}) + 1.$$

Find the value of $\left\lfloor \frac{a_{2014}}{2014} \right\rfloor$.

20. Let n be a positive integer such that, for each of the digits $0,\ 1, \ldots,\ 9$, there exists a factor of n ending in that digit. What is the smallest possible value of n?

21. Let $a_1, a_2, a_3, \cdots, a_{2001}, \cdots$ be an arithmetic progression such that $a_1^2 + a_{1001}^2 \leq 10$. Find the largest possible value of the following expression

$$a_{1001} + a_{1002} + a_{1003} + \cdots + a_{2001}.$$

22. It is given that w, a, b and c are positive integers that satisfy the equation

$$w! = a! + b! + c!.$$

Find the largest possible value of $w + a + b + c$.

23. In the triangle ABC, $AB = 63$ cm, $BC = 56$ cm, $CA = 49$ cm, M is the midpoint of BC, and the extension of AM meets the circumcircle ω of the triangle ABC at P. The circle through P and tangent to BC at M intersects ω at Q distinct from P. Find the length of MQ in centimetres.

24. Let M and C be two distinct points on the arc AB of a circle such that M is the midpoint of the arc AB. If D is the foot of the perpendicular from M onto the chord AC such that $AD = 100$ cm and $DC = 36$ cm, find the length of the chord BC in centimetres.

25. Let AD be the bisector of $\angle A$ of the triangle ABC. Let M and N be points on AB and AC, respectively such that $\angle MDA = \angle ABC$ and $\angle NDA = \angle ACB$. Let P be the intersection between AD and MN. Suppose $AD = 14$ cm. Find the value of $AB \times AC \times AP$ in cm^3.