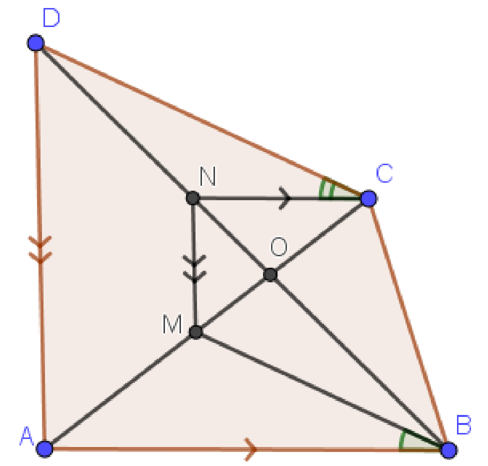
SMO Junior 2023 2nd Round (Solutions)

1. In a convex quadrilateral $ABCD$, the diagonals intersect at O, M and N are points on the segments OA and OD respectively. Suppose MN is parallel to AD and NC is parallel to AB. Prove that $\angle ABM = \angle NCD$.



Solution:

We claim $CD||BM$.

Since $CN||AB$, $\Delta CON \sim \Delta AOB$. Hence, $\frac{CO}{AO} = \frac{ON}{OB}$, giving $AO \cdot ON = CO \cdot OB$.

Since $MN||AD$, $\Delta MON \sim \Delta AOD$. Hence, $\frac{ON}{OD} = \frac{MO}{AO}$, giving $AO \cdot ON = MO \cdot OD$.

Equating, $CO \cdot OB = MO \cdot OD \Rightarrow \frac{CO}{OD} = \frac{MO}{OB}$. By SAS similarity, $\Delta COD \sim \Delta MOB$.

This implies $\angle DCO = \angle OMB$. Since we also have $\angle NCO = \angle OAB$ (from $CN||AB$), we conclude $\angle DCN = \angle DCO - \angle NCO = \angle OMB - \angle OAB = \angle ABM$ as desired.

2. What is the maximum number of integers that can be chosen from $1,2, \dots ,99$ so that the chosen integers can be arranged in a circle with the property that the product of every pair of neighbouring integers is a 3-digit integer.

Solution:

We claim there are at most 59 numbers.

Proof:

Clearly, 1 cannot be used in any arrangement.

Now call the 30 numbers $2, \dots ,31$ "small" and remaining numbers $32,33, \dots ,99$ "big". Note that since $32 \times 33 > 999$, no two big numbers can be adjacent in such an arrangement.

Since there are 30 small numbers, this implies there are at most 30 big numbers, otherwise two of them would be adjacent. Hence, there are at most 60 big numbers in total.

However, if there are exactly 60 numbers, then the small and big numbers must alternate. Yet among the big numbers, 31 can only be adjacent to 32 since $31 \times 33 = 1023 > 1000$. So 60 numbers is not achievable, and we have at most 59.

Construction:

Arrange the numbers 2 to 60 (in a circle) as follows:

$$2,60,3,59,4,58, \dots ,30,32,31.$$

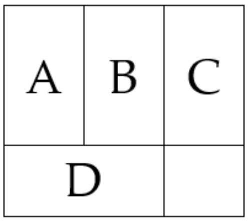
In this construction, each pair of adjacent numbers (other than (31,2)) clearly sums to either 62 or 63. Thus, their product ranges from $2 \times 60 = 120$ to $31 \times 32 = 992$, which are all 3-digit numbers.

However, $2 \times 31 = 62$ is not a 3-digit number, so we rectify this by simply swapping the numbers 2 and 4, giving us:

$$4,60,3,59,2,58, \dots ,30,32,31.$$

The new products formed are $4 \times 60 = 240, 3 \times 60 = 180, 3 \times 59 = 177$ and $4 \times 31 = 124$ which are all 3-digit numbers, so the construction is valid.

3. Define a domino to be a 1×2 rectangular block. A 2023×2023 square grid is filled with non-overlapping dominoes, leaving a single 1×1 gap. John then repeatedly slides dominoes into the gap; each domino is moved at most once. What is the maximum number of times that John could have moved a domino? (Example: In the 3×3 grid shown below, John could move 2 dominoes: D, followed by A.)



Solution:

We claim the maximum number of moves is $1012^2 - 1 = 1024143$.

Proof:

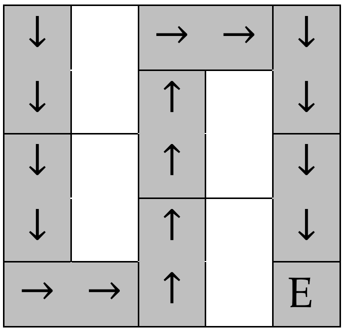
Label the squares in the grid (0,0) to (2022,2022).

Consider the position of the gap after a domino is moved. It is easy to see that the parity of the coordinates of the square will not change. Additionally, the same square cannot contain the gap twice, otherwise this implies that a domino was moved into that square, and was moved out of the square again later, which is disallowed.

Considering coordinates mod 2, in the 2023×2023 grid, there are 1012^2 squares labelled (0,0), $1012 \cdot 1011$ square labelled (0,1) or (1,0), and 1011^2 squares labelled (1,1). Hence, the number of moves is at most $1012^2 - 1$, if all the (0,0) squares are the 'gap square' at some point.

Construction:

This is attainable by connecting these squares in a 'snake' pattern, and sliding dominos accordingly; the remaining 1×2022 rectangles can be also tiled with dominos that do not move throughout. For example, here is the path for a 5×5 grid. Since each (0,0) mod 2 square is used exactly once, there must have been exactly $1012^2 - 1$ moves.



4. Two distinct 2-digit prime numbers p, q can be written one after the other in 2 different ways to form two 4-digit numbers. For example, 11 and 13 yield 1113 and 1311. If the two 4-digit numbers formed are both divisible by the average value of p and q, find all possible pairs (p, q).

Solution:

WLOG assume $p < q$.

The condition implies that $\frac{p+q}{2}$ is a factor of both $100p + q$ and $100q + p$.

Then $\frac{p+q}{2}$ is also a factor of $(100q + p) - 2 \cdot \frac{p+q}{2} = 99q$.

Since $\frac{p+q}{2} < \frac{q+q}{2} = q$ and q is prime, $\gcd\left(\frac{p+q}{2}, q\right) = 1$. Thus, $\frac{p+q}{2}$ divides 99.

Clearly, $12 = \frac{11+13}{2} \leq \frac{p+q}{2} \leq \frac{89+97}{2} = 93$. Hence, the only possible factor is 33, so $p + q = 66$ is the only possibility. Moreover, $100p + q = 99p + (p + q) = 99p + 66$ will clearly be divisible by $\frac{p+q}{2} = 33$. So any pair with $p + q = 66$ works.

Listing, the possible pairs are
(13,53), (19,47), (23,43), (29,37), (37,29), (43,23), (47,19) and (53,13).

5. Find all positive integers k such that there exists positive integers a, b such that
$$a^2 + 4 = (k^2 - 4)b^2.$$

Solution:

We claim that only $k = 3$ works. It is clear that for $k = 3$, $a = b = 1$ will satisfy the equation.

Since $a^2 < (k^2 - 4)b^2 < k^2b^2$, let $a = kb - c$ where c is a positive integer. Then substituting and simplifying, we get

$$k = \frac{4b^2 + 4 + c^2}{2bc}$$

Clearly, this implies c is even. So letting $c = 2x$, we get:

$$k = \frac{b^2 + x^2 + 1}{bx}.$$

(*Remark: It is a "well-known", but hard-to-prove fact that the only possible integer value of the RHS expression is 3. The proof uses Vieta jumping, and is reproduced below.*

Vieta Jumping has never been tested at the SMO Junior level; in fact, its only previous appearance in the SMO was in Senior 2nd Round 2022 Q4.)

Suppose there exists a $k \neq 3$ with such b, x. Let (b, x) be the pair with minimal sum such that $\frac{b^2+x^2+1}{bx} = k$. WLOG assume $x \geq b > 0$.

Then $x^2 - (kb)x + (b^2 + 1) = 0$.

Let x' be the 2nd root to this quadratic. By Vieta's,

$$x + x' = kb$$

$$xx' = b^2 + 1$$

Since $x' = kb - x, x'$ is an integer. Since $x' = \frac{b^2+1}{x}$, x' is positive. Hence, (b, x') is another pair which satisfies the equation, by the minimality assumption, $x' \geq x$.

Then $b^2 + 1 = xx' \geq x^2 \geq b^2$.

Clearly, $b^2 + 1$ is not a square. Thus, $x^2 = b^2$, implying that $x = b$.

Substituting back, we get $k = \frac{b^2+b^2+1}{b^2} = 2 + \frac{1}{b^2}$, which implies that the only integer combination is $b = 1, k = 3$, contradicting $k \neq 3$.