Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2023

Junior Section (Round 1)

Tuesday, 30 May 2023 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. In a certain company, one-third of the employees have a university degree. Half of the employees who have a university degree are men and 40% of the employees who do not have a university degree are women. If 102 of the employees are men, how many of the employees are women?

   (A) 58 (B) 78 (C) 98 (D) 108 (E) 118

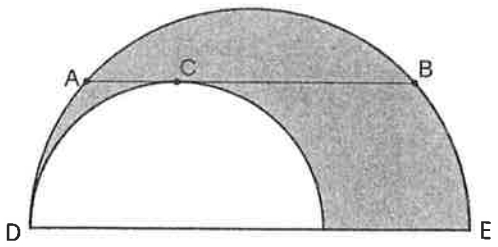
2. How many non-congruent triangles with integer side lengths have perimeter 7?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

3. The median and mean of five distinct numbers, $4, 7, 10, 11, N$, are equal. Find the sum of all possible values of N.

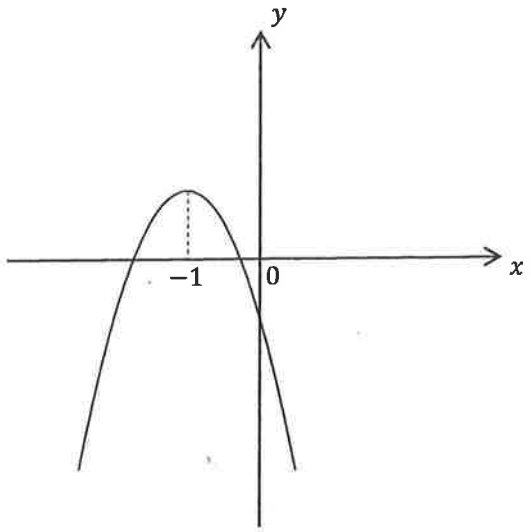
   (A) 18 (B) 21 (C) 26 (D) 29 (E) 35

4. The following diagram shows two semicircles whose diameters lie on the same line. AB is a chord of the larger semicircle that is tangent to the smaller semicircle at the point C and is parallel to the diameter DE of the larger semicircle. If |AB| = 16 cm, what is the area of the shaded region in cm^2?



   (A) 8π (B) 16π (C) 32π (D) 48π (E) 64π

5. The graph (not drawn to scale) of $y = ax^2 + bx + c$ with the maximum point at $x = -1$ is shown below. Which of the following must be true?



(I) $a - b + c > 0$ (II) $a + b - c > 0$ (III) $abc > 0$ (IV) $abc < 0$

(A) (I) and (IV)

(B) (II) and (IV)

(C) (I) and (III)

(D) (II) and (III)

(E) (I)

Short Questions

6. The product of the ages of three adults is 26390. Find the sum of their ages. (A person is an adult if he or she is at least 21 years old.)

7. Let n be a positive integer such that $n + 11$ is a factor of $n^2 + 121$. Find the largest possible value of n.

8. Find the largest integer less than or equal to $(3 + \sqrt{5})^3$.

9. The product of the two-digit number $\overline{x4}$ and the three-digit number $\overline{3yz}$ is 7656. Find the value of $x+y+z$.

10. If x and y are real numbers such that $x+y=12$ and $xy=10$, find the value of x^4+y^4.

11. Let a and b be two positive integers such that $a^2+4b^2 \leq 1105$. Find the largest possible value of the product ab.

12. Find the value of the integer n such that the following equation holds:

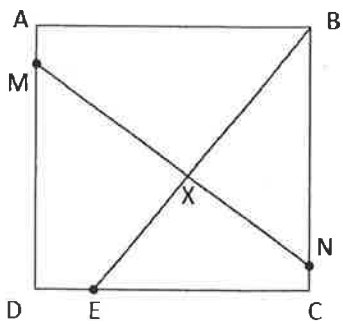
$$\frac{\sqrt{5}+n\sqrt{3}-2\sqrt{2}}{(\sqrt{5}+\sqrt{3})(\sqrt{3}-\sqrt{2})}=\sqrt{5}+\sqrt{2}.$$

13. The lengths of the sides of a triangle are $\log_{10} 12$, $\log_{10} 75$ and $\log_{10} n$ where n is a positive integer. Find the number of possible values for n.

14. Let x be a real number. What is the minimum value of the following expression?

$$\frac{20x^2+10x+3}{2x^2+x+1}$$

15. In the following diagram, ABCD is a square of side 16 cm. E lies on CD such that |DE| = 4 cm. M and N lie on AD and BC respectively such that MN is perpendicular to BE. X is the intersection of MN and BE. If |MX| = 11 cm and |BN| = x cm, what is the value of x?



16. Find the smallest positive integer k such that every subset of $\{1, 2, \ldots, 2023\}$ with exactly k integers contains three integers a, b, c with $a < b < c$ and $ab = c$.

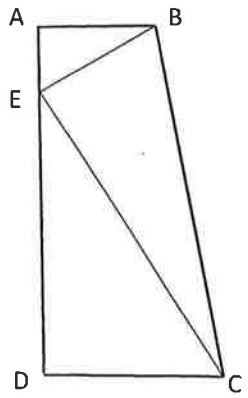
17. The difference between the least common multiple (LCM) and highest common factor (HCF) of a positive integer n and 18 is 627. Find the value of n.

18. The sum of all the interior angles except one of a convex polygon is 2023°. What is the number of sides of this polygon? (A polygon is convex if every interior angle is between 0° and 180°.)

19. If $\sqrt{19 - 8\sqrt{3}}$ is a root of the equation $x^2 - ax + b = 0$ where a and b are rational numbers, find the value of $a + b$.

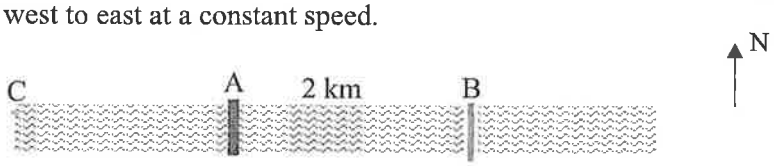
20. Find the smallest positive integer n such that the equation $x^2 + y^3 = n^4$ has a solution in positive integers x and y.

21. In the following diagram, AB is parallel to DC, |AB| = 6 cm, |AD| = 17 cm, |DC| = 10 cm and angle DAB = 90°. E lies on AD such that BE is perpendicular to EC. If the area of triangle BEC = k cm^2, what is the largest possible value of k?



22. The angle between the hour-hand and the minute-hand of a clock is denoted by $\theta°$ where $0 \leq \theta \leq 180$. If the clock now shows $3 : y$ (that is, 3 o'clock and y minutes) and $\theta = 175$, find the value of y, where y is an integer and $0 \leq y < 60$.

23. The diagram below shows a river with two bridges at A and B that are 2 km apart. The water flows from west to east at a constant speed.



A man rowed a boat upstream. He dropped his water bottle at A but he only noticed it at C after rowing for another 20 minutes upstream from A. He immediately turned back and rowed his boat downstream, and was able to pick up his water bottle at B. Assume that the man rowed his boat at a constant speed.

Find the speed of the water flowing in the river in km/h.

24. Four positive integers x, y, z and w satisfy the following equations:

$$xy + x + y = 104$$
$$yz + y + z = 146$$
$$zw + z + w = 524$$

If the product $xyzw = 2^7 \times 3^2 \times 5 \times 7$, find the value of $x + y + z + w$.

25. In the following diagram, ABC is a triangle. Points E, D and F lie on the side BC and divide the side into four equal parts. M is the midpoint of AB and CM intersects the line segments AE, AD and AF into the ratio $x : y : z : u$ where $x \geq y \geq z \geq u$ and x, y, z and u are integers. Find the least possible value of $x + y + z + u$.