Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2018

Junior Section (Round 1)

Wednesday, 30 May 2018 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

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Multiple Choice Questions

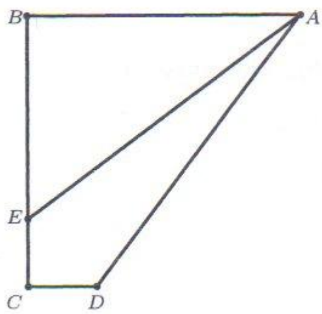
1. Which of the five numbers
$$\frac{9}{11}, \frac{19}{22}, \frac{29}{33}, \frac{39}{44} \text{ and } \frac{49}{55},$$
has the largest value?

(A) $\frac{9}{11}$ (B) $\frac{19}{22}$ (C) $\frac{29}{33}$ (D) $\frac{39}{44}$ (E) $\frac{49}{55}$

2. Find the value of
$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2018}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2017}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2018}\right)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2017}\right).$$

(A) $\frac{1}{2017}$ (B) $\frac{1}{2018}$ (C) $\frac{2017}{2018}$ (D) 1 (E) $\frac{2019}{2018}$

3. Given that $ABCD$ is a right-angled trapezium with $AB = BC$, $\angle ABC = \angle BCD = 90°$ and E is a point on BC such that $AE = AD$. If $AD = 10$ and $BE = 6$, find the length of DE.



(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 2 (D) $2\sqrt{2}$ (E) $2\sqrt{3}$

4. In a strange island, there are only two types of inhabitants: truth-tellers who only tell the truth and liars who only tell lies. One day, you meet two such inhabitants A and B. A said: "At least one of us is a liar." B kept silent. Which of the following must be true?

(A) Both A and B are truth-tellers
(B) Both A and B are liars
(C) A is a truth-teller and B is a liar
(D) A is a liar and B is a truth-teller
(E) Not enough information to decide

5. Suppose $ax + 5 = 0$ is satisfied by some x, where $-5 < x < 5$. Which one of the following conditions describes completely the range of a?

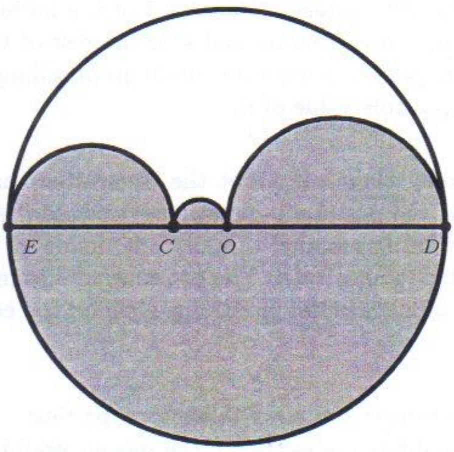
(A) $0 < a < 1$ (B) $a > 1$ or $a < -1$ (C) $a > 1$
(D) $a < -1$ (E) $a > 1$ or $-1 < a < 0$

Short Questions

6. There are 24 identical containers, each with a capacity of 12 litres. The containers contain some amount of water. If half of the containers are half full, a quarter of the containers are a quarter full, one-sixth of the containers are one-sixth full and one-twelfth of the containers are one-twelfth full, find the total amount of water (in litres) that are in the containers.

7. A quadruple (a, b, c, d) of positive integers is *balanced* if the mean, median and mode of a, b, c, d are equal. How many balanced quadruples (a, b, c, d) of positive integers are there that satisfy $a \le b \le c \le d$ and $a + b + c + d = 44$?

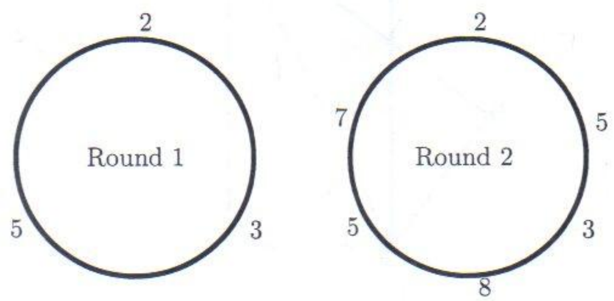
8. In the following diagram, ED is a diameter of the circle, EC, CO and OD are respective diameters of the three semi-circles. If $EC = 12$, $CO = 4$, $OD = 16$ and the area of the shaded region is $k\pi$, what is the value of k?



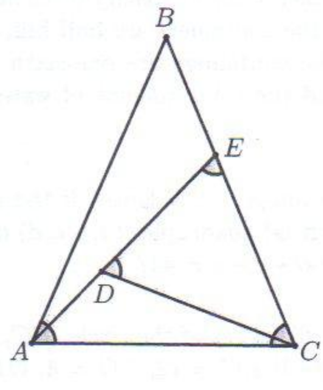
9. If $m^2 - 5m - 300 = 0$, find the value of

$$m^3 - 325m + 1503.$$

10. In Round 1 of a game, three numbers 2, 3 and 5 are written around a circle. In Round 2, the sum of every two adjacent numbers with respect to the circle is written between them. The following diagram shows the numbers after Rounds 1 and 2. Find the sum of all the numbers written around the circle after Round 5.



11. In the following diagram, if $AD = 2$, $AC = 6$, $\angle BAC = \angle BCA = \angle DEC = \angle CDE$, find the value of AB^2.

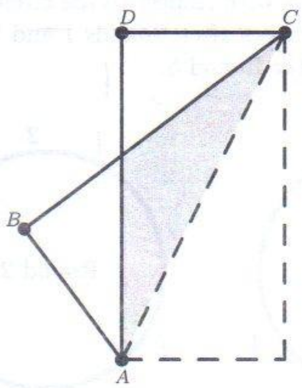


12. A business man bought p identical mobile phones at a total cost of m dollars, where p is a prime and m is a positive integer. He sold 2 of the mobile phones each at half of the cost price to a charitable organisation and sold the rest of the mobile phones in his shop at a profit of \$200 per phone. If his total profit from selling all these mobile phones was \$1800, find the least possible value of m.

13. Consider the x-y plane, where we adopt the convention that the positive x-direction is towards the right and the positive y-direction is upwards. An ant starts from the origin and crawls in the following manner: 2 units left, followed by 3 units up, followed by 4 units right, followed by 2 units down. The ant continues to move in the same pattern until it has moved a total of 2018 units. If the ant ends on the coordinate (m, n), what is the value of $m + n$?

14. A three-digit positive integer has the following properties:
(i) if its tens and ones digits are swapped, the integer would increase by 36;
(ii) if its hundreds and ones digits are swapped, the integer would decrease by 495.
Find the three-digit integer.

15. Let $ABCD$ be a rectangular sheet of paper with $AB = 12$ and $BC = 24$. If we fold the sheet of paper along the diagonal AC, there will be a overlapping region as shown in the diagram. Find the area of this overlapping region.

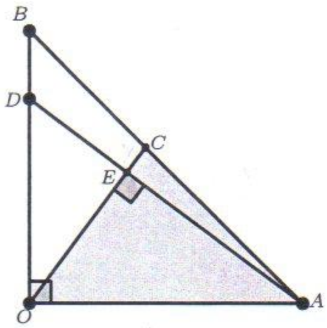


16. If $a - b = 3 + \sqrt{8}$ and $b - c = 3 - \sqrt{8}$, find the value of

$$a^2 + b^2 + c^2 - ab - bc - ac.$$

17. A teacher organised an outing for her students. The outing costs T dollars, regardless of how many students participated. This cost will be shared equally among the participants. If all her students were to participate, each student would pay a whole number of dollars. After the teacher announced how much each student would pay, there were 4 students who decided to withdraw. Without these four students, each of the remaining students would still pay a whole number of dollars. However, on the day of the outing, another 2 students were sick and absent. As these 2 sick students did not participate, each student who participated in the outing had to fork out an additional 3 dollars. If the teacher has between 24 and 40 students (inclusive), find T, the total cost of the outing in dollars.

18. In the following diagram, $\triangle AOB$ is an isosceles right-angled triangle with $OA = OB = 40$. Point D lies on the side OB such that $OD : DB = 3 : 1$. Point E is a point on AD such that OE is perpendicular to AD and OE extended meets AB at C. If $[OAC]$ denotes the area of $\triangle OAC$, find the value of $7[OAC]$.



19. Find the value of

$$\sqrt{9^2 + 19} + \sqrt{19^2 + 39} + \sqrt{29^2 + 59} + \sqrt{39^2 + 79} + \cdots + \sqrt{639^2 + 1279}.$$

20. If x is a positive real number that satisfies $x^2 = \sqrt{3}$, evaluate

$$\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1}.$$

21. If x is a real number such that $x^3 + \frac{1}{x^3} = 18$, determine the value of $\left(x + \frac{1}{x}\right)^2$.

22. When expressed as a fraction in the lowest terms,

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{99 \times 100 \times 101} = \frac{m}{n}.$$

Find the value of $m + n$.

23. If α, β and γ satisfy

$$(x-\alpha)(x-\beta)(x-\gamma) = x^3 - 18x^2 - 122x + 161,$$

find the value of $\alpha^2 + \beta^2 + \gamma^2$.

24. How many integers x satisfy

$$x^2 + 2017x \leq 2018x + 2019?$$

25. Determine the coefficient of x in the following expansion:

$$1 - (1-x) + (1-x)^2 - (1-x)^3 + (1-x)^4 - \cdots - (1-x)^{2017}.$$