Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2017

Junior Section (Round 1)

Tuesday, 30 May 2017 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x. *For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

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Multiple Choice Questions

1. Among the five numbers $\frac{23}{44}$, $\frac{24}{45}$, $\frac{25}{46}$, $\frac{26}{47}$ and $\frac{27}{48}$, which one has the smallest value?

   (A) $\frac{23}{44}$ (B) $\frac{24}{45}$ (C) $\frac{25}{46}$ (D) $\frac{26}{47}$ (E) $\frac{27}{48}$

2. Let a and b be real numbers satisfying $\frac{1}{a} < \frac{1}{b} < 0$. Which of the following is incorrect?

   (A) $|a| < |b|$ (B) $a > b$ (C) $a + b < ab$ (D) $a^3 > b^3$ (E) $a^2 > b^2$

3. How many ways can the letters of the word "IGLOO" be arranged?

   (A) 4 (B) 5 (C) 30 (D) 60 (E) 120

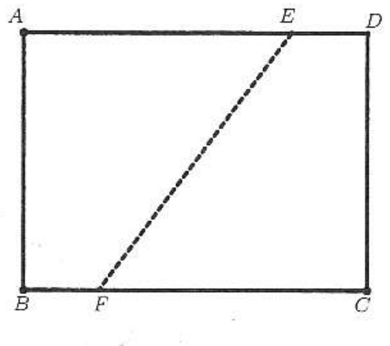
4. Jenny and Mary received identical fruit baskets, each containing 3 apples, 4 oranges and 2 bananas. Assuming that both Jenny and Mary randomly picked a fruit from their own basket, what is the probability that they both picked an apple?

   (A) $\frac{1}{9}$ (B) $\frac{1}{6}$ (C) $\frac{1}{3}$ (D) $\frac{2}{3}$ (E) None of the above

5. A cylinder has base radius r and height $7r$. If a sphere has the same surface area as the cylinder, find the ratio of the volume of the cylinder to the volume of the sphere.

   (A) $\frac{3}{4}$ (B) $\frac{4}{3}$ (C) $\frac{4}{21}$ (D) $\frac{21}{32}$ (E) 7

6. Let $ABCD$ be a rectangular sheet of paper with $AB = 6$ and $BC = 8$. We can fold the paper along the crease line EF so that point C coincides with point A. Find the length of the resulting line segment AF.



   (A) $\frac{25}{4}$ (B) $\frac{13}{2}$ (C) $\frac{27}{4}$ (D) 7 (E) None of the above

7. Given three consecutive positive integers, which of the following is a possible value for the difference of the squares of the largest and the smallest of these three integers?

   (A) 91 (B) 92 (C) 93 (D) 94 (E) 95

8. You have 30 rods of length 5, 30 rods of length 17 and 30 rods of length 19. Using each rod at most once, how many non-congruent triangles can you form?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

9. Let a and b be positive integers. If the highest common factor of a and b is 6 and the lowest common multiple of a and b is $2^3 3^4 5^5$, how many possible values are there for a?

(A) 2 (B) 4 (C) 8 (D) 14 (E) 16

10. If x and y are non-zero real numbers satisfying $x + y = 2$ and

$$\frac{(1-y)^2}{y} + \frac{(1-x)^2}{x} = -4,$$

find the value of xy.

(A) 1 (B) -1 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$ (E) $\sqrt{2}$

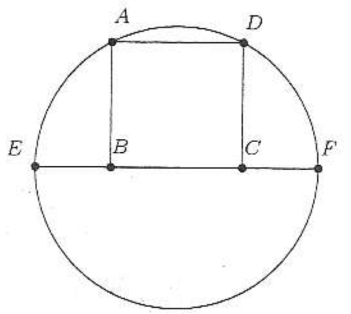
Short Questions

11. An n-sided polygon has two interior angles of sizes $94°$ and $51°$. The remaining interior angles are all equal in size. If $4 \le n \le 20$, determine the value of n.

12. Find the number of multiples of 7 in the sequence $80, 81, 82, \ldots, 2016, 2017$.

13. A list of six positive integers has a unique mode of 4, median of 6 and mean of 8. Find the largest possible integer in the list.

14. In the diagram, EF is a diameter of the circle and $ABCD$ is a square with points B and C on EF and points A and D on the circle. If $AB = 17\sqrt{5}$, find the length of EF.



15. Find the remainder when

$$2017^1 + 2017^2 + \cdots + 2017^{2017}$$

is divided by 9.

16. Assume that

$$(x+2)^{2017}+(x+2)^{2016}+\cdots+(x+2)^{2000}=a_{2017}x^{2017}+a_{2016}x^{2016}+\cdots+a_1x+a_0.$$

Find the value of the following expression:

$$(a_0-a_1)+(a_2-a_3)+(a_4-a_5)+\cdots+(a_{2016}-a_{2017}).$$

17. If $x=\sqrt{2017}+1$, find the value of

$$x^3-(2+\sqrt{2017})x^2+(1+2\sqrt{2017})x-\sqrt{2017}.$$

18. Let ABC be a triangle, D be a point on AC such that $AD=DC$ and E be a point on BC such that $BE=2EC$. Let F be the intersection of BD and AE. If the area of triangle ABC is 100, find the area of triangle ADF.

19. Find the largest integer from 1 to 100 which has exactly 3 positive integer divisors. For example, the only positive divisors of 4 are 1, 2 and 4.

20. Let a,b and c be positive integers such that $a^2+bc=257$ and $ab+bc=101$. Determine the value of abc.

21. In a trapezium $ABCD$, AD is parallel to BC and points E and F are the midpoints of AB and DC respectively. If

$$\frac{\text{Area of } AEFD}{\text{Area of } EBCF}=\frac{\sqrt{3}+1}{3-\sqrt{3}},$$

and the area of triangle ABD is $\sqrt{3}$, find the area of the trapezium $ABCD$.

22. Let $\frac{5+\sqrt{23}}{2}=a+b$, where a is a positive integer and b is a real number satisfying $0\le b<1$. Evaluate $a^3+(3+\sqrt{23})b$.

23. Let a,b and c be the three solutions of the equation $x^3-4x^2+5x-6=0$. Determine the value of $a^2+b^2+c^2+3abc$.

24. Let a be an integer such that both $a+79$ and $a+2$ are perfect squares. Find the largest possible value of a.

25. Determine the number of integers x which satisfy the following inequality.

$$x^2+2015x+1<2017x+2017^2.$$

26. If every root of the polynomial x^2+4x-5 is also a root of the polynomial $2x^3+9x^2+bx+c$, find the value of b^2+c^2.

27. Let m be the minimum value of the quadratic curve $y = x^2 - 4ax + 5a^2 - 3a$, where the value m depends on a. If $0 \le a \le 6$, find the maximum possible value of m.

28. Let a, b, c, d and e be five consecutive positive integers where e is the largest. If $b+c+d$ is a perfect square and $a+b+c+d+e$ is a perfect cube, find the least possible value of e.

29. Find the value of a_1 if

$$x^5 = a_5(x-1)^5 + a_4(x-1)^4 + a_3(x-1)^3 + a_2(x-1)^2 + a_1(x-1) + a_0.$$

30. Let a and b be positive real numbers satisfying $a + b = 10$. Find the largest possible value of

$$\sqrt{10a+15} + \sqrt{10b+13}.$$

31. Find the value of

$$\sqrt{1 + 2017\sqrt{1 + 2016\sqrt{1 + 2015\sqrt{1 + 2014\sqrt{1 + 2013 \times 2011}}}}}.$$

32. Find the largest possible value of m such that the polynomial $x^2 + (2m-1)x + (m-6)$ has two real roots x_1 and x_2 satisfying $x_1 \le -1$ and $x_2 \ge 1$.

33. If one of the integers is removed from the first N consecutive integers $1, 2, 3, \ldots, N$, the resulting average of the remaining integers is $\frac{525}{13}$. Find N.

34. Amongst the fractions

$$\frac{1}{175}, \frac{2}{175}, \frac{3}{175}, \ldots, \frac{174}{175},$$

there are some which can be reduced to a fraction with a smaller denominator such as $\frac{5}{175} = \frac{1}{35}$, and there are some that cannot be reduced further like $\frac{1}{175}$. Find the sum of all the fractions which cannot be reduced further.

35. The number of seashells collected by 13 boys and n girls is $n^2 + 10n - 18$. If each child collects exactly the same number of seashells, determine the value of n.