


SMO 2011 Junior Q16:

16. Let $\lfloor x \rfloor$ be the greatest integer smaller than or equal to x . How many solutions are there to the equation $x^3 - \lfloor x^3 \rfloor = (x - \lfloor x \rfloor)^3$ on the interval $[1, 20]$?

LJ's solution:

<p>oneplusone ☆☆☆☆☆  Posts: 1109 Location: Doofensmirtz Evil Incorporated, Singapore</p>	<p style="text-align: right;">Posted: Jun 05, 2011, 11:45 pm • # 3</p> <p>Let $\lfloor x \rfloor = a$, and $b = x - a$. Then $0 \leq b < 1$. Now we sub it into the equation, we get $a^3 + 3a^2b + 3ab^2 = \lfloor x^3 \rfloor$. Note that this is true if and only if $3a^2b + 3ab^2$ is an integer. Note that $0 \leq 3a^2b + 3ab^2 < 3a^2 + 3a$, so for any given a, there are exactly $3a^2 + 3a$ values of b that satisfies the equation, since we can set b such that $3a^2b + 3ab^2$ takes any integer value from 0 to $3a^2 + 3a - 1$. So the total number of solutions in x is</p> $\sum_{n=1}^{19} (3n^2 + 3n) + 1 = 7981$ <p>Signatures are for noobs</p>
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Refer to below related solution from a Chinese Mathematical Olympiad Book:

(P91, 奥林匹克数学高一分册+钱展望+朱华伟+著,湖北教育出版社 2002,ISBN:7-5351-3143-3)

In this question, data range is $[1, n]$, In SMO Junior Q16, the data range is $[1, 20]$, so correct answer is $n^3 - n + 1 = 20^3 - 20 + 1 = 7981$. LJ's answer is correct.

例4 设 n 为一正整数,则在区间 $[1, n)$ 内有多少个实数 x , 满足方程

$$x^3 - [x^3] = (x - [x])^3. \quad \textcircled{1}$$

解 设 $x = m + r$, 其中 $m = [x]$, $r = \{x\}$. 则 $1 \leq m \leq n - 1, 0 \leq r < 1$. 代入①, 得

$$(m + r)^3 - [(m + r)^3] = (m + r - [m + r])^3,$$

即
$$m^3 + 3m^2r + 3mr^2 = [(m + r)^3]. \quad \textcircled{2}$$

②说明 $m^3 + 3m^2r + 3mr^2$ 是一个整数. 当 r 从 0 增大到 1 时, $m^3 + 3m^2r + 3mr^2$ 从 m^3 增大到 $m^3 + 3m^2 + 3m$. 即

$$m^3 \leq m^3 + 3m^2r + 3mr^2 < m^3 + 3m^2 + 3m,$$

所以, 对于 $0 \leq r < 1, m^3 + 3m^2r + 3mr^2$ 可以取

$$m^3 + 3m^2 + 3m - m^3 = (m + 1)^3 - m^3 - 1$$

个整数值.

由于 $1 \leq m \leq n - 1$, 所以, 满足原方程的 x 的个数为

$$\begin{aligned} \sum_{m=1}^{n-1} [(m + 1)^3 - m^3 - 1] &= n^3 - 1 - (n - 1) \\ &= n^3 - n. \end{aligned}$$

SMO 2011 Junior Q19:

19. Let a, b, c, d be real numbers such that

$$\begin{cases} a^2 + b^2 + 2a - 4b + 4 = 0, \\ c^2 + d^2 - 4c + 4d + 4 = 0. \end{cases}$$

Let m and M be the minimum and the maximum values of $(a - c)^2 + (b - d)^2$, respectively.

What is $m \times M$?

Rewrite two equations as below:

$$\begin{cases} (a + 1)^2 + (b - 2)^2 = 1^2 \\ (c - 2)^2 + (d + 2)^2 = 2^2 \end{cases}$$

Obviously, (a, b) is a point A on the first circle with center $(-1, 2)$ and radius 1, (c, d) is a point B on the second circle with center $(2, -2)$ and radius 2. Recall $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$,

Hence, m is the square of the minimum distance between A and B, M is the square of the maximum distance between A and B. the distance between two centers of these circles is 5.

$$\text{Thus, } m = (5 - 2 - 1)^2 = 4, M = (5 + 2 + 1)^2 = 64$$

$$m \times M = 4 \times 64 = 256$$

SMO 2011 Junior Q26:

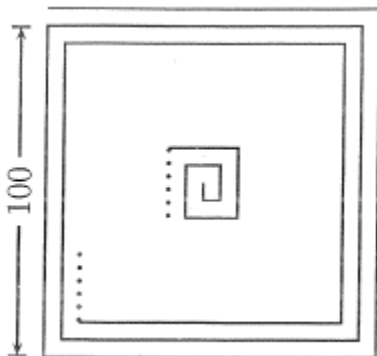
26. How many ways are ^{there} ~~three~~ to put 7 identical apples into 4 identical packages so that each package has at least one apple?

Since 7 apples are identical, there are only 3 ways to put them into 4 identical packages. Answer is 3.

In the SMO 2011 solution book, the answer 350 is for 7 distinct apples.

SMO 2011 Junior Q28:

28. Find the length of the spirangle in the following diagram, where the gap between adjacent parallel lines is 1 unit.



Starting point is wrong, left hand side is unable to reach 100. (it should be odd number), question is wrong.