SMO 2011 Junior Q16:

16. Let $\lfloor x \rfloor$ be the greatest integer smaller than or equal to x. How many solutions are there to the equation $x^3 - \lfloor x^3 \rfloor = (x - \lfloor x \rfloor)^3$ on the interval $[1, 20]$?

LJ's solution:

oneplusone

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Posted: Jun 05, 2011, 11:45 pm • # 3

Let $\lfloor x \rfloor = a$, and $b = x - a$. Then $0 \le b < 1$. Now we sub it into the equation, we get $a^3 + 3a^2b + 3ab^2 = \lfloor x^3 \rfloor$. Note that this is true if and only if $3a^2b + 3ab^2$ is an integer. Note that $0 \le 3a^2b + 3ab^2 < 3a^2 + 3a$, so for any given a, there are exactly $3a^2 + 3a$ values of b that satisfies the equation, since we can set b such that $3a^2b + 3ab^2$ takes any integer value from 0 to $3a^2 + 3a - 1$. So the total number of solutions in x is

$$\sum_{n=1}^{19} (3n^2 + 3n) + 1 = 7981$$

Signatures are for noobs

Refer to below related solution from a Chinese Mathematical Olympiad Book:

(P91, 奥林匹克数学高一分册+钱展望+朱华伟+著,湖北教育出版社 2002,ISBN:7-5351-3143-3)

In this question, data range is [1,n), In SMO Junior Q16, the data range is [1,20], so correct answer is $n^3 - n + 1 = 20^3 - 20 + 1 = 7981$. LJ's answer is correct.

例 4 设 n 为一正整数，则在区间$[1,n)$内有多少个实数 x，满足方程

$$x^3-[x^3]=(x-[x])^3. \qquad ①$$

解 设 $x=m+r$，其中 $m=[x]$，$r=\{x\}$. 则 $1\leq m\leq n-1, 0\leq r<1$. 代入①，得

$$(m+r)^3-[(m+r)^3]=(m+r-[m+r])^3,$$

即
$$m^3+3m^2r+3mr^2=[(m+r)^3]. \qquad ②$$

②说明 $m^3+3m^2r+3mr^2$ 是一个整数. 当 r 从 0 增大到 1 时，$m^3+3m^2r+3mr^2$从 m^3 增大到 m^3+3m^2+3m. 即

$$m^3\leq m^3+3m^2r+3mr^2<m^3+3m^2+3m,$$

所以，对于 $0\leq r<1$，$m^3+3m^2r+3mr^2$ 可以取

$$m^3+3m^2+3m-m^3=(m+1)^3-m^3-1$$

个整数值.

由于 $1\leq m\leq n-1$，所以，满足原方程的 x 的个数为

$$\sum_{m=1}^{n-1}[(m+1)^3-m^3-1]=n^3-1-(n-1)$$
$$=n^3-n.$$

SMO 2011 Junior Q19:

19. Let a, b, c, d be real numbers such that

$$\begin{cases} a^2+b^2+2a-4b+4=0, \\ c^2+d^2-4c+4d+4=0. \end{cases}$$

Let m and M be the minimum and the maximum values of $(a-c)^2+(b-d)^2$, respectively. What is $m\times M$?

Rewrite two equations as below:

$$\begin{cases}(a+1)^2+(b-2)^2=1^2\\(c-2)^2+(d+2)^2=2^2\end{cases}$$

Obviously, (a, b) is a point A on the first circle with center (-1, 2) and radius 1, (c, d) is a point B on the second circle with center (2, -2) and radius 2. Recall $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$,

Hence, m is the square of the minimum distance between A and B, M is the square of the maximum distance between A and B. the distance between two centers of these circles is 5.

Thus, $m = (5-2-1)^2 = 4$, $M = (5+2+1)^2 = 64$

$m \times M = 4 \times 64 = 256$

SMO 2011 Junior Q26:

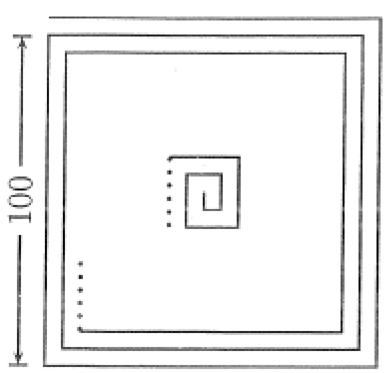
26. How many ways are ~~three~~ there to put 7 identical apples into 4 identical packages so that each package has at least one apple?

Since 7 apples are identical, there are only 3 ways to put them into 4 identical packages. Answer is 3.

In the SMO 2011 solution book, the answer 350 is for 7 distinct apples.

SMO 2011 Junior Q28:

28. Find the length of the spirangle in the following diagram, where the gap between adjacent parallel lines is 1 unit.



Starting point is wrong, left hand side is unable to reach 100. (it should be odd number), question is wrong.