This file contains the problems, suggested for solving on the Russian national mathematical competitions (final part).

I've posted this stuff in a number of articles in rec.puzzles. But I have got many requests for the missing parts. So I have decided to put this material here, having provided it with the answers on the common questions.

==================collected articles start======================

I'm going to send some problems from the book Vasil'ev N.B, Egorov A.A. "The problems of the All-Soviet–Union mathematical competitions",–Moscow.:Nauka. 1988 ISBN 5–02–013730–8. (in Russian).

Those problems were submitted for the solving on the competition between the pupils of 8, 9, or 10 forms for 4 hours. So they do not contain something of the advanced topics, — all of them can be solved by a schoolboy. They can not go out of the common school plan bounds.

Most of the problems are original and the book contains all the necessary references. I am not going to translate all the book, so I shall not send the solutions. Please, accept those messages as they are, to say more exactly – as I can. I have to do my job, and this is hobby only, but nevertheless, that should be enjoyable to solve those problems.

> "Nobody can embrace the unembraceable."
> Kozma Prutkov.
> (beginning of the XIX c.)

May be those postings are just a harassment, but I hope, that most of You will not only enjoy the problems solving, but will be able to use them in Your work with the students.

> "Zeal overcomes everything."
> "Sometimes zeal overcomes even the common sense."
> Kozma Prutkov.

There are some wonderful books in Russian, that have not been translated into English yet, for example,

"Problems of the Moscow mathematical competitions",–compiled by G.A.Galperin, A.K.Tolpygo.,–Moscow, Prosveshchenie, 1986.

A.A.Leman "Collection of Moscow mathematical competitions problems", –Moscow, Prosveshchenie, 1965.

==============answers on the common questions============

WHAT IS THE AGE OF THE PARTICIPANTS?

The Russian pupils start studying being 6–7 years old, so the pupils of the 8th form are about 14.

WHO PARTICIPATE?

The competition is held in 3 – 4 stages
1. At school – if there are many volunteers.
2. Subregional – if the region is big enough.
3. Regional (in some regions, as in Moscow, Leningrad=Sankt–Petersburg, Sverdlovsk = Yekaterinburg or Novosibirsk they are even more interesting and more difficult).
4. Final part, considered in the report.

WHAT ARE THE BEST AND AVERAGE RESULTS?

The winners (2–5) usually give the perfect solution of all the problems with some shortages.

My personal experience refers to that times, when there were two days of the final competition. Than the winners solved all the problems of two days except one problem.

It is very difficult to speak about the average level, because it depends very much on a region, but most of the participants of that time solved at least one problem. The problem is not only the difficulties in the problems themselves, but also in the shortage of time. They successfully solved the problems before the official explanation two days later.

MAY I USE THOSE PROBLEMS IN THE SCHOOL PROJECTS?

You don't need MY permission for using those problems. As concerns the copyright, the usage of all the information in the non-commercial purposes was never restricted in Russia if it is not related to the state security. Moreover, the spirit of the competition encourages everybody to distribute those problems in order to enhance the mathematical culture of the pupils.

And You only are able to decide whether Your students can solve them.

ARE THERE THE TRANSLATED SOLUTIONS OF THOSE PROBLEMS ELSEWHERE?

Sorry, there exists no complete translation of the cited book. Besides, the solutions (in comparison with the problems themselves) belong to the authors, and the translation without their explicit permission would be their copyright violation.

============collected articles continue===================

So... First one — Moscow, 1961.

```
form
    8  001   002   003   004   005a
    9  006a  007   008   009   010
   10  011   012   007   006b  005b
```

001.

```
+-------------+-------------+
|             |             |
|             |             |
+-------+-----+----+---------+
|       |          |         |
|       |          |         |
+-------+----------+---------+
```

Given a figure, containing 16 segments. You should prove that there is no curve, that intersect each segment exactly once. The curve may be not closed, may intersect itself, but it is not allowed to touch the segments or to pass through the vertices.

002.

Given a rectangle ABCD with AC length e and four circles centers A, B, C, D and radii a, b, c, d respectively, satisfying $a+c=b+d<e$. Prove you can inscribe a circle inside the quadrilateral whose sides are the two outer common tangents to the circles center A and C, and the two outer common tangents to the circles center B and D.

003.

Prove that among 39 sequential natural numbers there always is a number with the sum of its digits divisible by 11.

004.

Given a table 4x4.

a) Find, how 7 stars can be put in its fields in such a way, that erasing of two arbitrary lines and two columns will always leave at list one of the stars.
b) Prove that if there are less than 7 stars, You can always find two columns and two rows, that if You erase them, no star will remain in the table.

005.

a) Given a quadruple (a, b, c, d) of positive reals, transform to the new quadruple (ab, bc, cd, da). Repeat arbitarily many times. Prove that you can never return to the original quadruple unless $a=b=c=d=1$.
b) Given n a power of 2, and an n–tuple $(a_1, a_2, \dots, a_n)$ transform to a new n–tuple $(a_1a_2, a_2a_3, \dots, a_{n-1}a_n, a_na_1)$. If all the members of the original n–tuple are 1 or –1, prove that with sufficiently many repetitions you obtain all 1s.

006.

a) A and B move clockwise with equal angular speed along circles center P and Q respectively. C moves continuously so that AB=BC=CA. Establish C's locus and speed.
b) ABC is an equilateral triangle and P satisfies AP=2, BP=3. Establish the maximum possible value of CP.

007.

Given an m x n array of real numbers. You may change the sign of all numbers in a row or of all numbers in a column. Prove that by repeated changes you can obtain an array with all row and column sums non-negative.

008.

Given $n<1$ points, some pairs joined by an edge (an edge never joins a point to itself). Given any two distinct points you can reach one from the other in just one way by moving along edges. Prove that there are $n–1$ edges.

009.
Given any natural numbers m, n and k. Prove that we can always find relatively prime natural numbers r and s such that rm+sn is a multiple of k.

010.
A and B play the following game with N counters. A divides the counters into 2 piles, each with at least 2 counters. Then B divides each pile into 2 piles, each with at least one counter. B then takes 2 piles according to a rule which both of them know, and A takes the remaining 2 piles. Both A and B make their choices in order to end up with as many counters as possible.
There are 3 possibilities for the rule: *R1* B takes the biggest heap (or one of them if there is more than one) and the smallest heap (or one of them if there is more than one). *R2* B takes the two middling heaps (the two heaps that A would take under *R1*). *R3* B has the choice of taking either the biggest and smallest, or the two middling heaps.
For each rule, how many counters will A get if both players play optimally?

011.
Given three arbitary infinite sequences of natural numbers, prove that we can find unequal natural numbers m, n such that for each sequence the mth member is not less than the nth member.

012.
120 unit squares are arbitarily arranged in a 20 x 25 rectangle (both position and orientation is arbitary). Prove that it is always possible to place a circle of unit diameter inside the rectangle without intersecting any of the squares.

The second competition — Moscow, 1962.

form					
8	013	014	015	016	017
9	018	019	020	021	017
10	022	023	024	025	026

013.
ABCD is any convex quadrilateral. Construct a new quadrilateral as follows. Take A' so that A is the midpoint of DA'; similarly, B' so that B is the midpoint of AB'; C' so that C is the midpoint of BC'; and D' so that D is the midpoint of CD'. Show that the area of A'B'C'D' is five times the area of ABCD.

014.
Given a fixed circle C and a line L throught the center O of C. Take a variable point P on L and let K be the circle center P through O. Let T be the point where a common tangent to C and K meets K. What is the locus of T?

015.
Given integers $a_0, a_1, \ldots , a_{100}$, satisfying:

$$a_1>a_0,\ a_1>0, \text{ and}$$
$$a_{r+2}=3\ a_{r+1} – 2\ a_r \text{ for } r=0, 1, \ldots , 98.$$

Prove $a_{100} > 2^{99}$.

016.
Prove that there are no integers a, b, c, d such that the polynomial ax^3+bx^2+cx+d equals 1 at x=19 and 2 at x=62.

017.
Given an n x n array of numbers. n is odd and each number in the array is 1 or –1. Prove that the number of rows and columns containing an odd number of –1s cannot total n.

018.
Given the lengths AB and BC and the fact that the medians to those two sides are perpendicular, construct the triangle ABC.

019.
Given four positive real numbers a, b, c, d such that abcd=1, prove that:

$$a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \geq 10.$$

020.
Given a fixed regular pentagon ABCDE with side 1. Let M be an arbitary point inside or on it. Let the distance from M to the closest vertex be r_1, to the next closest be r_2 and so on, so that the distances from M to the five vertices satisfy $r_1 \leq r_2 \leq r_3 \leq r_4 \leq r_5$.
Find:
(a) the locus of M which gives r_3 the minimum possible value;
(b) the locus of M which gives r_3 the maximum possible value.

021.
Given a number with 1998 digits which is divisible by 9. Let x be the sum of its digits, let y be the sum of the digits of x, and z the sum of the digits of y. Find z.

022.
AB=BC and M is the midpoint of AC. H is chosen on BC so that MH is perpendicular to BC. P is the midpoint of MH. Prove that AH is perpendicular to BP.

023.
The triangle ABC satisfies $0 \leq AB \leq 1 \leq BC \leq 2 \leq CA \leq 3$. What is the maximum area it can have?

024.
Given unequal integers x, y, z prove that $(x-y)^5 + (y-z)^5 + (z-x)^5$ is divisible by $5(x-y)(y-z)(z-x)$.

025.
Given $a_0, a_1, \ldots, a_n$, satisfying $a_0 = a_n = 0$, and and $a_{k-1} - 2a_k + a_{k+1} \geq 0$ for k=0, 1, ... , n–1. Prove that all the numbers are negative or zero.

026.
Given two sets of positive numbers with the same sum. The first set has m numbers and te second n. Prove that you can find a set of less than m+n positive numbers which can be arranged to part fill an m x n array, so that the row and column sums are the two given sets.

The third competition — Moscow, 1963.

form					
8	027	028	029a	030	031a
9	032	033	034	031b	028
10	035	036	037	029b	028
11	038	028	039	040	029b

027.
Given 5 circumferences, every four of them have a common point. Prove that there exist a point that belongs to all five circumferences.

028.
Eight men had participated in the chess tournament. (Each meets each; draws are allowed, giving 1/2 of pont; winner gets 1.) Everyone has different number of points. The second one has got as many points as the the four weakest together. What was the result of the play between the third prizer and the chess–player that have occupied the seventh place?

029.
a) Each diagonal of the quadrangle halves its area. Prove that it is a parallelogram.
b) Three main diagonals of the hexagon halve its area. Prove that they intersect in one point.

030.
Natural numbers a and b are relatively prime.
Prove that the greatest common divisor of (a+b) and (a^2+b^2) is either 1 or 2.

031.
Given two fixed points A and B .The point M runs along the circumference containing A and B. K is the middle of the segment [MB]. [KP] is a perpendicular to the line (MA).
a) Prove that all the possible lines (KP) pass through one point.
b) Find the set of all the possible points P.

032.
Find the smallest value x such that, given any point inside an equilateral triangle of side 1, we can always choose two points on the sides of the triangle, collinear with the given point and a distance x apart.

033.
(a) A 6 x 6 board is tiled with 2 x 1 dominos. Prove that we can always divide the board into two rectangles each of which is tiled separately (with no domino crossing the dividing line).
(b) (b) Is this true for an 8 x 8 board?

034.
Given a set of n different positive reals $\{a_1, a_2, \dots , a_n\}$. Take all possible non-empty subsets and form their sums. Prove we get at least n(n+1)/2 different sums.

035.
Given a triangle ABC. Let the line through C parallel to the angle bisector of B meet the angle bisector of A at D, and let the line through C parallel to the angle bisector of A meet the angle bisector of B at E. Prove that if DE is parallel to AB, then CA=CB.

036.
Given the endless arithmetic progression with the positive integer members. One of those is an exact square. Prove that the progression contain the infinite number of the exact squares.

037.
Can we label each vertex of a 45–gon with one of the digits 0, 1, ... , 9 so that for each pair of distinct digits i, j one of the 45 sides has vertices labeled i, j?

038.
Find such real p, q, a, b, that for all x an equality is held:

$$(2x-1)^{20} - (ax+b)^{20} = (x^2+px+q)^{10}.$$

039.
We place labeled points on a circle as follows. At step 1, take two points at opposite ends of a diameter and label them both 1. At step n>1, place a point at the midpoint of each arc created at step n–1 and label it with the sum of the labels at the two adjacent points. What is the total sum of the labels after step n?
For example, after step 4 we have: 1, 4, 3, 5, 2, 5, 3, 4, 1, 4, 3, 5, 2, 5, 3, 4.

040.
Given an isosceles triangle, find the locus of the point P inside the triangle such that the distance from P to the base equals the geometric mean of the distances to the sides.

The 4–th competition — Moscow, 1964.

form					
8	041	042	043	044	045a
9	041	046	047	048	049
10	050	051	045ab	052	053
11	054	055	052	053	054

041.
In the triangle ABC, the length of the altitude from A is not less than BC, and the length of the altitude from B is not less than AC. Find the angles.

042.
Prove that for no natural m a number m(m + 1) is a power of an integer.

043.
Reduce each of the first billion natural numbers (billion = 10^9) to a single digit by taking its digit sum repeatedly. Do we get more 1s than 2s?

044.
Given n odd and a set of integers $a_1, a_2, \dots, a_n$, derive a new set $(a_1 + a_2)/2, (a_2 + a_3)/2, \dots, (a_{n-1} + a_n)/2, (a_n + a_1)/2$. However many times we repeat this process for a particular starting set we always get integers. Prove that all the numbers in the starting set are equal. For example, if we started with 5, 9, 1, we would get 7, 5, 3, and then 6, 4, 5, and then 5, 4.5, 5.5. The last set does not consist entirely of integers.

045.
a) The convex hexagon ABCDEF has all angles equal. Prove that AB – DE = EF – BC = CD – FA.
b) Given six lengths $a_1, \dots, a_6$ satisfying $a_1 - a_4 = a_5 - a_2 = a_3 - a_6$, show that you can construct a hexagon with sides $a_1, \dots, a_6$ and equal angles.

046.
Find integer solutions (x,y) of the equation (1964 times "√t"):

$$\sqrt{t}(x + \sqrt{t}(x + \sqrt{t}(\dots(x + \sqrt{t}(x))\dots))) = y.$$

047.
ABCD is a convex quadrilateral. A' is the foot of the perpendicular from A to the diagonal BD, B' is the foot of the perpendicular from B to the diagonal AC, and so on. Prove that A'B'C'D' is similar to ABCD.

048.
Find all natural numbers n such that n^2 does not divide n!.

049.
Given a lattice of regular hexagons. A bug crawls from vertex A to vertex B along the edges of the hexagons, taking the shortest possible path (or one of them). Prove that it travels a distance at least AB/2 in one direction. If it travels exactly AB/2 in one direction, how many edges does it traverse?

050.
A circle center O is inscribed in ABCD (touching every side). Prove that angle AOB + angle COD equals 180 degrees.

051.
The natural numbers a, b, n are such that for every natural number k not equal to b, b – k divides $a - k^n$. Prove that $a = b^n$.

052.
How many (algebraically) different expressions can we obtain by placing parentheses in $a_1/a_2/ \dots /a_n$?

053.
What is the smallest number of tetrahedrons into which a cube can be partitioned?

054.
(a) Find the smallest square with last digit not 0 which becomes another square by the deletion of its last two digits.
(b) Find all squares, not containing the digits 0 or 5, such that if the second digit is deleted the resulting number divides the original one.

055.
A circle is inscribed in ABCD. AB is parallel to CD, and BC = AD. The diagonals AC, BD meet at E. The circles inscribed in ABE, BCE, CDE, DAE have radius r_1, r_2, r_3, r_4 respectively. Prove that $1/r_1 + 1/r_3 = 1/r_2 + 1/r_4$.

The fifth competition — Moscow, 1965.

form					
8	056a	057	058	059	060
9	061	062	063	064	065
10	056b	066	067a	068a	069
11	063	067b	070	068b	071

056.

a) Each of $x_1, \dots, x_n$ is –1, 0 or 1. What is the minimal possible value of the sum of all $x_i x_j$ with $1 \le i < j \le n$?

b) Is the answer the same if the x_i are real numbers satisfying $0 \le |x_i| \le 1$ for $1 \le i \le n$?

057.

Two players have a 3 x 3 board. 9 cards, each with a different number, are placed face up in front of the players. Each player in turn takes a card and places it on the board until all the cards have been played. The first player wins if the sum of the numbers in the first and third rows is greater than the sum in the first and third columns, loses if it is less, and draws if the sums are equal. Which player wins and what is the winning strategy?

058.

A circle is circumscribed about the triangle ABC. X is the midpoint of the arc BC (on the opposite side of BC to A), Y is the midpoint of the arc AC, and Z is the midpoint of the arc AB. YZ meets AB at D and YX meets BC at E. Prove that DE is parallel to AC and that DE passes through the center of the inscribed circle of ABC.

059.

Bus numbers have 6 digits, and leading zeros are allowed. A number is considered lucky if the sum of the first three digits equals the sum of the last three digits. Prove that the sum of all lucky numbers is divisible by 13.

060.

The beam of a lighthouse on a small rock penetrates to a fixed distance d. As the beam rotates the extremity of the beam moves with velocity v. Prove that a ship with speed at most v/8 cannot reach the rock without being illuminated.

061.

A group of 100 people is formed to patrol the local streets. Every evening 3 people are on duty. Prove that you cannot arrange for every pair to meet just once on duty.

062.

A tangent to the inscribed circle of a triangle drawn parallel to one of the sides meets the other two sides at X and Y. What is the maximum length XY, if the triangle has perimeter p?

063.

The n^2 numbers x_{ij} satisfy the n^3 equations: $x_{ij} + x_{jk} + x_{ki} = 0$. Prove that we can find numbers $a_1, \dots, a_n$ such that $x_{ij} = a_i - a_j$.

064.

Can 1965 points be arranged inside a square with side 15 so that any rectangle of unit area placed inside the square with sides parallel to its sides must contain at least one of the points?

065.
Given n real numbers $a_1, a_2, \ldots, a_n$, prove that you can find n integers $b_1, b_2, \ldots, b_n$, such that the sum of any subset of the original numbers differs from the sum of the corresponding b_i by at most $(n + 1)/4$.

066.
A tourist arrives in Moscow by train and wanders randomly through the streets on foot. After supper he decides to return to the station along sections of street that he has traversed an odd number of times. Prove that this is always possible. [In other words, given a path over a graph from A to B, find a path from B to A consisting of edges that are used an odd number of times in the first path.]

067.
a) A committee has met 40 times, with 10 members at every meeting. No two people have met more than once at committee meetings. Prove that there are more than 60 people on the committee.
b) Prove that you cannot make more than 30 subcommittees of 5 members from a committee of 25 members with no two subcommittees having more than one common member.

068.
Given two relatively prime natural numbers r and s, call an integer *good* if it can be represented as mr + ns with m, n non-negative integers and *bad* otherwise. Prove that we can find an integer c, such that just one of k, c – k is good for any k. How many bad numbers are there?

069.
A spy–plane circles point A at a distance 10 km with speed 1000 km/h. A missile is fired towards the plane from A at the same speed and moves so that it is always on the line between A and the plane. How long does it take to hit?

070.
Prove that the sum of the lengths of the edges of a polyhedron is at least 3 times the greatest distance between two points of the polyhedron.

071.
An alien moves on the surface of a planet with speed not exceeding u. A spaceship searches for the alien with speed v. Prove the spaceship can always find the alien if $v>10u$.

The sixth competition — Voronezh, 1966.

form					
8	072	073a	074	075a	076
9	077	073b	075b	078	079
10,11	075b	080	081	082	083

072.
There are an odd number of soldiers on an exercise. The distance between every pair of soldiers is different. Each soldier watches his nearest neighbour. Prove that at least one soldier is not being watched.

073.
a) B and C are on the segment AD with AB = CD. Prove that for any point P in the plane: $PA + PD \geq PB + PC$.
b) Given four points A, B, C, D on the plane such that for any point P on the plane we have $PA + PD \geq PB + PC$. Prove that B and C are on the segment AD with AB = CD.

074.
Can both $x^2 + y$ and $x + y^2$ be squares for x and y natural numbers?

075.
A group of children are arranged into two equal rows. Every child in the back row is taller than the child standing in front of him in the other row. Prove that this remains true if each row is rearranged so that the children increase in height from left to right.

076.
A rectangle ABCD is drawn on squared paper with its vertices at lattice points and its sides lying along the gridlines. AD = k AB with k an integer. Prove that the number of shortest paths from A to C starting out along AD is k times the number starting out along AB.

077.
Given non-negative real numbers $a_1, a_2, \ldots, a_n$, such that $a_{i-1} \le a_i \le 2a_{i-1}$ for $i = 2, 3, \ldots, n$. Show that you can form a sum $s = b_1a_1 + \ldots + b_na_n$ with each b_i +1 or –1, so that $0 \le s \le a_1$.

078.
Prove that you can always draw a circle radius A/P inside a convex polygon with area A and perimeter P.

079.
A graph has at least three vertices. Given any three vertices A, B, C of the graph we can find a path from A to B which does not go through C. Prove that we can find two disjoint paths from A to B.
[A graph is a finite set of vertices such that each pair of distinct vertices has either zero or one edges joining the vertices. A path from A to B is a sequence of vertices $A_1, A_2, \ldots, A_n$ such that $A=A_1$, $B=A_n$ and there is an edge between A_i and A_{i+1} for $i = 1, 2, \ldots, n-1$. Two paths from A to B are disjoint if the only vertices they have in common are A and B.]

080.
Given a triangle ABC. Suppose the point P in space is such that PH is the smallest of the four altitudes of the tetrahedron PABC. What is the locus of H for all possible P?

081.
Given 100 points on the plane. Prove that you can cover them with a collection of circles whose diameters total less than 100 and the distance between any two of which is more than 1. [The distance between circles radii r and s with centers a distance d apart is the greater of 0 and $d - r - s$.]

082.
The distance from A to B is d kilometers. A plane P is flying with constant speed, height and direction from A to B. Over a period of 1 second the angle PAB changes by α degrees and the angle PBA by □ degrees. What is the minimal speed of the plane?

083.
Two players alternately choose the sign for one of the numbers 1, 2, ... , 20. Once a sign has been chosen it cannot be changed. The first player tries to minimize the final absolute value of the total and the second player to maximize it. What is the outcome (assuming both players play perfectly)?
Example: the players might play successively: 1, 20, –19, 18, –17, 16, –15, 14, –13, 12, –11, 10, –9, 8, –7, 6, –5, 4, –3, 2. Then the outcome is 12. However, in this example the second player played badly!

"Where is the beginning of that end,
that ends the beginning?"
Kozma Prutkov.

"If You see a title 'buffalo'
on the elephant's cage
— don't believe to Your eyes!"
Kozma Prutkov.

The name has been changed — the numeration was restarted.

The first competition — Tbilisi, 1967.

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form
     8  084a  085a  086a  087   088
     9  087b  086a  085b  084b  088
    10  090   086b  091   092   093
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084.
In the acute–angled triangle ABC, AH is the longest altitude (H lies on BC), M is the midpoint of AC, and CD is an angle bisector (with D on AB).
a) If AH ≤ BM, prove that the angle ABC ≤ 60.
b) If AH = BM = CD, prove that ABC is equilateral.

085.
The digits of a natural number are rearranged and the resultant number is added to the original number. Prove that the answer cannot be 99 ... 9 (1999 nines).
The digits of a natural number are rearranged and the resultant number is added to the original number to give 10^{10}. Prove that the original number was divisible by 10.

086.
Four lighthouses are arbitarily placed in the plane. Each has a stationary lamp which illuminates an angle of 90 degrees. Prove that the lamps can be rotated so that at least one lamp is visible from every point of the plane.

087.
a) Can you arrange the numbers 0, 1, ... , 9 on the circumference of a circle, so that the difference between every pair of adjacent numbers is 3, 4 or 5? For example, we can arrange the numbers 0, 1, ... , 6 thus: 0, 3, 6, 2, 5, 1, 4.
b) The same question, but about the numbers 0,1,...,13.

088.
Prove that there exists a number divisible by 5^{1000} with no zero digit.

089.
Find all integers x, y satisfying $x^2 + x = y^4 + y^3 + y^2 + y$.

090.
What is the maximum possible length of a sequence of natural numbers x_1, x_2, x_3, ... such that $x_i \le 1998$ for $i \ge 1$, and $x_i = |x_{i-1} - x_{i-2}|$ for $i \ge 3$.

091.
"KING–THE SUICIDER"
499 white rooks and a black king are placed on a 1000 x 1000 chess board. The rook and king moves are the same as in ordinary chess, except that taking is not allowed and the king is allowed to remain in check. No matter what the initial situation and no matter how white moves, the black king can always:
(a) get into check (after some finite number of moves);

(b) move so that apart from some initial moves, it is always in check after its move;
(c) move so that apart from some initial moves, it is always in check (even just after white has moved).
Prove or disprove each of (a) – (c).

092.
ABCD is a unit square. One vertex of a rhombus lies on side AB, another on side BC, and a third on side AD. Find the area of the set of all possible locations for the fourth vertex of the rhombus.

093.
A natural number k has the property that if k divides n, then the number obtained from n by reversing the order of its digits is also divisible by k. Prove that k is a divisor of 99.

The second competition — Leningrad, 1968.

form	first day					second day				
8	094	095	096	097	098	105a	106	107	108	109
9	099	100	101	097	102	110	111	105a	108	109
10	103	095	104	097	096	105b	112	113	114	109

094.
An octagon has equal angles. The lengths of the sides are all integers. Prove that the opposite sides are equal in pairs.

095.
Which is greater: 31^{11} or 17^{14}? [No calculators allowed!]

096.
A circle radius 100 is drawn on squared paper with unit squares. It does not touch any of the grid lines or pass through any of the lattice points. What is the maximum number of squares it can pass through?

097.
In a group of students, 50 speak English, 50 speak French and 50 speak Spanish. Some students speak more than one language. Prove it is possible to divide the students into 5 groups (not necessarily equal), so that in each group 10 speak English, 10 speak French and 10 speak Spanish.

098.
Prove that:

$$2/(x^2 – 1) + 4/(x^2 – 4) + 6/(x^2 – 9) + ... + 20/(x^2 – 100) =$$
$$11/((x – 1)(x + 10)) + 11/((x – 2)(x + 9)) + ... + 11/((x – 10)(x + 1)).$$

099.
The difference between the longest and shortest diagonals of the regular n–gon equals its side. Find all possible n.

100.
The sequence a_n is defined as follows: $a_1 = 1$, $a_{n+1} = a_n + 1/a_n$ for $n \geq 1$. Prove that $a_{100} > 14$.

101.
Given two acute–angled triangles ABC and A'B'C' with the points O and O' inside. Three pairs of the perpendiculars are drawn:

- $[OA_1]$ to the side [BC], $[O'A'_1]$ to the side [B'C'],
- $[OB_1]$ to the side [AC], $[O'B'_1]$ to the side [A'C'],

- [OC_1] to the side [AB], [$O'C'_1$] to the side [A'B'];

it is known that

- [OA_1] is parallel to the [O'A'],
- [OB_1] is parallel to the [O'B'],
- [OC_1] is parallel to the [O'C']

and the following products are equal:

$$|OA_1|*|O'A'| = |OB_1|*|O'B'| = |OC_1|*|O'C'|.$$

Prove that

- [$O'A'_1$] is parallel to the [OA],
- [$O'B'_1$] is parallel to the [OB],
- [$O'C'_1$] is parallel to the [OC]

and $|O'A'_1|*|OA| = |O'B'_1|*|OB| = |O'C'_1|*|OC|$.

102.
Prove that you can represent an arbitrary number not exceeding n! (n–factorial; n!= 1*2*3*...*n) as a sum of k different numbers (k≤n) that are divisors of n!.

103.
Given a triangle ABC, point D on [AB], E on [AC]; |AD| = |DE| = |AC| , |BD| = |AE| , DE is parallel to BC. Prove that the length |BD| equals to the side of a right decagon (ten–angle) inscribed in a circle with the radius R=|AC|.

104.
Three spheres are built so that the edges [AB], [BC], [AD] of the tetrahedron ABCD are their respective diameters. Prove that the spheres cover all the tetrahedron.

105.
a)

+	-	+	+
+	+	+	+
+	+	+	+
+	+	+	+

The fields of the square table 4x4 are filled with the "+" or "–" signs. You are allowed to change the signs simultaneously in the whole row, column, or diagonal to the opposite sign. That means, for example, that You can change the sign in the corner square, because it makes a diagonal itself. Prove that You will never manage to obtain a table containing pluses only.

b) The fields of the square table 8x8 are filled with the "+" or signs except one non-corner field with "–". You are allowed to change thesigns simultaneously in the whole row, column, or diagonal to the opposite sign. That means, for example, that You can change the sign in the corner field, because it makes a diagonal itself. Prove that You will never manage to obtain a table containing pluses only.

106.
Medians divide the triangle onto 6 smaller ones. 4 of the circles inscribed in those small ones are equal. Prove that the triangle is equilateral.

107.
Prove that the equation $x^2 + x + 1 = py$ has solution (x,y) for the infinite number of simple p.

108.

Each of the 9 referees on the figure skating championship estimates the program of 20 sportsmen by assigning him a place (from 1 to 20). The winner is determined by adding those numbers. (The less is the sum – the higher is the final place). It was found, that for the each sportsman, the difference of the places, received from the different referees was not greater than 3.
What can be the maximal sum for the winner?

109.
Two finite sequences $a_1,a_2,...,a_n$; $b_1,b_2,...,b_n$ are just rearranged sequence 1, 1/2, ... , 1/n.
$a_1+b_1 \geq a_2+b_2 \geq ... \geq a_n+b_n$.
Prove that for every m ($1 \leq m \leq n$) $a_m+a_n \geq 4/m$.

110.
There is scales on the teacher's table. There is a set of weighs on the scales, and there are some pupils' names (may be more than one) on the every weigh. A pupil entering the classroom moves all the weight with his name to another side of the scales. Prove that You can let in such a subset of the pupils, that the scales will change its position.

111.
The city is a rectangle divided onto squares by m streets coming from the West to the East and n streets coming from the North to the South. There are militioners (policemen) on the streets but not on the crossroads. They watch the certain automobile, moving along the closed route, marking the time and the direction of its movement. Its trace is not known in advance, but they know, that it will not pass over the same segment of the way twice.
What is the minimal number of the militioners providing the unique determination of the route according to their reports?

112.
The circle inscribed in the triangle ABC touches the side [AC] in the point K. Prove that the line connecting the middle of the [AC] side with the centre of the circle halves the [BK] segment.

113.
The sequence $a_1,a_2,...,a_n$ satisfies the following conditions:

$$a_1=0,\ |a_2|=|a_1+1|,\ ...,\ |a_n|=|a_{n-1}+1|.$$

Prove that $(a_1+a_2+...+a_n)/n \geq -1/2$.

114.
Given a quadrangle ABCD. The lengths of all its sides and diagonals are the rational numbers. Let O be the point of its diagonals intersection. Prove that |AO| – the length of the [AO] segment is also rational.

The third competition — Kiev, 1969.

form	first day			second day		
8	115	116	117	122	123	124a
9	118	119	115	124	125	126
10	119	120	121	125	126	128

115.
The point E lies on the base [AD] of the trapezoid ABCD. The triangles ABE, BCE and CDE perimeters are equal. Prove that $|BC| = |AD|/2$.

116.
There is a wolf in the centre of a square field, and four dogs in the corners. The wolf can easily kill one dog, but two dogs can kill thewolf. The wolf can run all over the field, and the dogs — along the fence (border) only. Prove that if the dog's speed is 1.5 times morethan the wolf's, than the dogs can prevent the wolf escaping.

117.
Given a finite sequence of 0's and 1's with two properties:

- if you chose five sequential digits in one place and in the second place, those will be two different binary numbers. (Some last digits of the first number may be included as the first digits in the second.)
- if You add 0 or 1 either from the left or from the right side, the previous property will not be held.

Prove that the first four digits of that sequence coincide with the last four.

118.
Given positive numbers a,b,c,d. Prove that the set of inequalities

$$a+b<c+d;$$
$$(a+b)(c+d)<ab+cd;$$
$$(a+b)cd<ab(c+d)$$

contain at least one wrong.

119.
For what minimal natural a the polynomial $ax^2 + bx + c$ with the integer c and b has two different positive roots both less than one.

120.
Given natural n. Consider all the fractions $1/(pq)$, where p and q are relatively prime;

$$0<p<q\le n \; ; \; p+q>n.$$

Prove that the sum of all such a fractions equals to 1/2.

121.
Given n points in the three dimensional space such, that the arbitrary triangle with the vertices in three of those points contains an angle greater than 120 degrees. Prove that You can rearrange them to make a polyline (unclosed) with all the angles between the sequent links greater than 120 degrees.

122.
Find four different three–digit decimal numbers starting with the same digit, such that their sum is divisible by three of them.

123.
Every city in the certain state is connected by airlines with no more than with three other ones, but one can get from every city to every oher city changing a plane once only or directly. What is the maximal possible number of the cities?

124.
Given a pentagon with all equal sides.
a) Prove that there exist such a point on the maximal diagonal, that every side is seen from it inside a right angle.
/* I mean that the side AB is seen from the point C inside an arbitrary angle that is greater or equal than angle ACB. – VAP */
b) Prove that the circles build on its sides as on the diameters cannot cover the pentagon entirely.

125.
Given an equation $x^3 + ?x^2 + ?x + ? = 0$. First player substitutes an integer on the place of one of the interrogative marks, than the same do the second with one of the two remained marks, and, finally, the first puts the integer instead of the last mark. Explain how can the first provide the existence of three integer roots in the obtained equation. (The roots may coincide.)

126.
20 football teams participate in the championship. What minimal number of the games should be played to provide the property: from the three arbitrary teams we can find at least on pair that have already met in the championship.

127.
Let h_k be an apothem of the right k–angle inscribed into a circle with radius R. Prove that $(n + 1)h_{n+1} – nh_n > R$.

128.
Prove that for the arbitrary positive $a_1, a_2, \ldots, a_n$ the following inequality is held

$$a_1/(a_2+a_3)+a_2/(a_3+a_4)+....a_{n-1}/(a_n+a_1)+a_n/(a_1+a_2)>-n/4$$

The 4–th competition — Simferopol, 1970.

form	first day					second day		
8	129	130	131	132	133a			
9	134	135	133b	136	137			
10	138	139	133b	136	140	141	142	143

129.
Given a circle, its diameter [AB] and a point C on it. Build (with thehelp of compasses and ruler) two points X and Y, that are symmetric with respect to (AB) line, such that (YC) is orthogonal to (XA).

130.
The product of three positive numbers equals to one, their sum is strictly greater than the sum of the inverse numbers. Prove that one and only one of them is greater than one.

131.
How many sides of the convex polygon can equal its longest diagonal?

132.
The digits of the 17–digit number are rearranged in the reverse order. Prove that at list one digit of the sum of the new and the initial number is even.

133.
a) A castle is equilateral triangle with the side of 100 metres. It is divided onto 100 triangle rooms. Each wall between the rooms is 10 metres long and contain one door. You are inside and are allowed to pass

through every door not more than once. Prove that You can visit not more than 91 room (not exiting the castle).

b) Every side of the triangle is divided onto k parts by the lines parallel to the sides. And the triangle is divided onto k^2 small triangles. Let us call the "chain" such a sequence of triangles, that every triangle in it is included only once, and the consecutive triangles have the common side. What is the greatest possible number of the triangles in the chain?

134.
Given five segments. It is possible to builda triangle of every subset of three of them. Prove that at least one of those triangles is acute–angled.

135.
The bisector [AD], the median [BM] and the height [CH] of the acute–angled triangle ABC intersect in one point. Prove that the angle BAC is greater than 45 degrees.

136.
Given five n–digit binary numbers. For each two numbers their digits coincide exactly on m places. There is no place with the common digit for all the five numbers.
Prove that $2/5 \le m/n \le 3/5$.

137.
Prove that from every set of 200 integers You can choose a subset of 100 with the total sum divisible by 100.

138.
Given triangle ABC, middle M of the side [BC], the centre O of the inscribed circle. The line (MO) crosses the height AH in the point E.
Prove that the distance |AE| equals the inscribed circle radius.

139.
Prove that for every natural number k there exists an infinite set of such natural numbers t, that the decimal notation of t does not contain zeroes and the sums of the digits of the numbers t and kt are equal.

140.
Two equal rectangles are intersecting in 8 points. Prove that the common part area is greater than the half of the rectangle's area.

141.
All the 5–digit numbers from 11111 to 99999 are written on the cards. Those cards lies in a line in an arbitrary order.
Prove that the resulting 444445–digit number is not a power of two.

142.
All natural numbers containing not more than n digits are divided onto two groups. The first contains the numbers with the even sum of the digits, the second — with the odd sum. Prove that if $0 < k < n$ than the sum of the k–th powers of the numbers in the first group equals to the sum of the k–th powers of the numbers in the second group.

143.
The vertices of the right n–angle are marked with some colours (each vertex — with one colour) in such a way, that the vertices of one colour represent the right polygon.
Prove that there are two equal ones among the smaller polygons.

The 5–th competition — Riga, 1971.

form	first day					second day		
8	144	145a	146a	147		152ab	153	154
9	144	145a	148	147	146b	156abc	152c	155
10	149	145b	150	147	151b	156	157	158

144.
Prove that for every natural n there exists a number, containing only digits "1" and "2" in its decimal notation, that is divisible by 2^n (n–th power of two).

145.

a) Given a triangle $A_1A_2A_3$ and the points
   B_1 and D_2 on the side $[A_1A_2]$,
   B_2 and D_3 on the side $[A_2A_3]$,
   B_3 and D_1 on the side $[A_3A_1]$.
   If You build parallelograms $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ and $A_3B_3C_3D_3$, the lines (A_1C_1), (A_2C_2) and (A_3C_3), will cross in one point O.
   Prove that if $|A_1B_1| = |A_2D_2|$ and $|A_2B_2| = |A_3D_3|$, than $|A_3B_3| = |A_1D_1|$.

b) Given a convex polygon $A_1A_2 \dots A_n$ and the points
   B_1 and D_2 on the side $[A_1A_2]$,
   B_2 and D_3 on the side $[A_2A_3]$,
   B_n and D_1 on the side $[A_nA_1]$.
   If You build parallelograms $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ ... , $A_nB_nC_nD_n$, the lines (A_1C_1), (A_2C_2), ..., (A_nC_n), will cross in one point O.
   Prove that $|A_1B_1|*|A_2B_2|*\dots*|A_nB_n| = |A_1D_1|*|A_2D_2|*\dots*|A_nD_n|$.

146.

a) A game for two. The first player writes two rows of ten numbers each, the second under the first. He should provide the following property: if number b is written under a, and d — under c, then a + d = b + c. The second player has to determine all the numbers. He is allowed to ask the questions like "What number is written in the x place in the y row?" What is the minimal number of the questions asked by the second player before he founds out all the numbers?

b) There was a table mxn on the blackboard with the property: if You chose two rows and two columns, then the sum of the numbers in the two opposite vertices of the rectangles formed by those lines equals the sum of the numbers in two another vertices. Some of the numbers are cleaned. but it is still possible to restore all the table. What is the minimal possible number of the remaining numbers?

147.
Given an unite square and some circles inside. Radius of each circle is less than 0.001, and there is no couple of points belonging to thedifferent circles with the distance between them 0.001 exactly.
Prove that the area, covered by the circles is not greater than 0.34.

148.
The volumes of the water containing in each of three big enough containers are integers. You are allowed only to relocate some times from one container to another the same volume of the water, that the destination already contains. Prove that You are able to discharge one of the containers.

149.
Prove that if the numbers p_1, p_2, q_1, q_2 satisfy the condition

$$(q_1 - q_2)^2 + (p_1 - p_2)(p_1q_2 - p_2q_1) < 0,$$

then the square polynomials

$$x^2 + p_1x + q_1 \text{ and } x^2 + p_2x + q_2$$

have real roots, and between the roots of each there is a root of another one.

150.
The projections of the body on two planes are circles.
Prove that they have the same radius.

151.
Some numbers are written along the ring. If inequality $(a–d)(b–c) < 0$ is held for the four arbitrary numbers in sequence a,b,c,d, You have to change the numbers b and c places.
Prove that You will have to do this operation finite number of times.

152.
a) Prove that the line dividing the triangle onto two polygons with equal perimeters and equal areas passes through the centre of the inscribed circle.
b) Prove the same statement for the arbitrary polygon outscribed around the circle.
c) Prove that all the lines halving its perimeter and area simultaneously, intersect in one point.

153.
Given 25 different positive numbers. Prove that You can choose two of them such, that none of the other numbers equals neither to the sum nor to the difference between the chosen numbers.

154.
a) The vertex A_1 of the right 12–angle (dodecagon) $A_1A_2...A_{12}$ is marked with "–" and all the rest — with "+". You are allowed to change the sign simultaneously in the 6 vertices in succession. Prove that is impossible to obtain dodecagon with A_2 marked with "–" and the rest of the vertices — with "+".
b) Prove the same statement if it is allowed to change the signs not in six, but in four vertices in succession.
c) Prove the same statement if it is allowed to change the signs in three vertices in succession.

155.
N unit squares on the infinite sheet of cross–lined paper are painted with black colour. Prove that You can cut out the finite number of square pieces and satisfy two conditions:
– all the black squares are contained in those pieces.
– the area of black squares is not less than 1/5 and not greater than 4/5 of every piece area.

The task for the tenth form was as follows:
"You are given three serious problems. Try to investigate at least one, but to obtain as many results, as You can. At the end of Your work make a sort of resume, showing the main proved facts, challenged examples and the hypotheses that seem to be true ..."
/* this form of competition was never repeated later – it had required too much efforts from those who checked the works */

156.
A cube with the edge of length n is divided onto n^3 unit ones. Let us choose some of them and draw three lines parallel to the edges through their centres. What is the least possible number of the chosen small cubes necessary to make those lines cross all the smaller cubes?

a) Find the answer for the small n (n = 2,3,4).
b) Try to find the answer for n = 10.
c) If You can not solve the general problem, try to estimate that value from the upper and lower side.
d) Note, that You can reformulate the problem in such a way: Consider all the triples (x_1,x_2,x_3), where x_i can be one of the integers 1,2,...,n. What is the minimal number of the triples necessary to provide the property: for each of the triples there exist the chosen one, that differs only in one coordinate. Try to find the answer for the situation with more than three coordinates, for example, with four.

157.

a) Consider the function $f(x,y) = x^2 + xy + y^2$. Prove that for the every point (x,y) there exist such integers (m,n), that $f((x–m),(y–n)) \le 1/2$.

b) Let us denote with g(x,y) the least possible value of the f((x–m),(y–n)) for all the integers m,n. The statement a) was equal to the fact $g(x,y) \le 1/2$. Prove that in fact, $g(x,y) \le 1/3$. Find all the points (x,y), where g(x,y)=1/3.

c) Consider function $f_a(x,y) = x^2 + axy + y^2$ $(0 \le a \le 2)$. Find any c such that $g_a(x,y) \le c$. Try to obtain the closest estimation.

158.

```
 1    2          1    2
 |    |          |    |
 v    v          v    v
+-+---+-+      +-+---+-+
| |   | |      | |   | |
|  \ /  |      | |   | |
|   X   |      | |   | |
|  / \  |      | |   | |
| |   | |      | |   | |
+-+---+-+      +-+---+-+
 |    |          |    |
 V    V          V    V
 1    2          1    2
```

The switch with two inputs and two outputs can be in one of two different positions. In the left part of the picture a) the first input is connected with the second output and we can denote this as

```
1 2
| |
V V
2 1
```

and the position in the right part of the picture will be denoted as

```
1 2
| |
V V
1 2
```

fig.a)

```
   1    2    3
   |    |    |
   V    V    |
+-+---+-+    |
|       |    |
|       |    |
+-+---+-+    |
   |    |    |
   |    V    V
   |  +-+---+-+
   |  |        |
      | |        |
      |  +-+---+-+
   |    |    |
   V    V    |
+-+---+-+    |
|       |    |
|       |    |
+-+---+-+    |
   |    |    |
   V    V    V
   1    2    3
       f
```

(fig.b)

The scheme on the picture b) is universal in that sense that changing the state of the element switches You can obtain all the six connections, i.e.

```
1 2 3   1 2 3   1 2 3   1 2 3   1 2 3   1 2 3
| | |   | | |   | | |   | | |   | | |   | | |
V V V   V V V   V V V   V V V   V V V   V V V
1 2 3   1 3 2   2 1 3   2 3 1   3 1 2   3 2 1
```

(Check it. Note, that the total number of the states is 2^3 = 8, because each element can be in two positions.)

a) Try to build the universal scheme for 4 inputs and 4 outputs, that can provide all of 24 possible connections.

b) What is the minimal number of the element switches for such a scheme?
Let us call a scheme with n inputs and n outputs n–universal, if it can provide all n! possible connections of n inputs with n outputs.

c) Here is the scheme (picture c) with 8 inputs and 8 outputs, where A and B are 4–universal. Prove that it is 8–universal.

```
  1        2        3        4             5        6        7        8
  |        |        |        |             |        |        |        |
  V  V     V  V     V  V     V  V       V  V     V  V     V  V     V  V
+-+--+-+ +-+--+-+ +-+--+-+ +-+--+-+   +-+--+-+ +-+--+-+ +-+--+-+ +-+--+-
+
|      | |      | |      | |      |   |      | |      | |      | |
|
|      | |      | |      | |      |   |      | |      | |      | |
|
+-+--+-+ +-+--+-+ +-+--+-+ +-+--+-+   +-+--+-+ +-+--+-+ +-+--+-+ +-+--+-
+
  |    \   |    \   |    \   |    \   /    |   /    |   /    |   /    |
  |     \  |     \  |     \  |     \ /     |  /     |  /     |  /     |
  |      \ |      \ |      \ |      X      | /      | /      | /      |
 ......................................................................

  |/       |/       |/       |/           \|       \|       \|       \|
+-+--------+--------+--------+----+   +----+--------+--------+--------+-
+
|                                 |   |
|
|              A                  |   |               B
|
|                                 |   |
|
+-+--------+--------+--------+----+   +----+--------+--------+--------+-
+
  |\       |\       |\       |\           /|       /|       /|       /|
 ......................................................................

  |    /   |    /   |    /   |    /   \    |   \    |   \    |   \    |
  |   /    |   /    |   /    |   /     \   |    \   |    \   |    \   |
  V  V     V  V     V  V     V  V       V  V     V  V     V  V     V  V
+-+--+-+ +-+--+-+ +-+--+-+ +-+--+-+   +-+--+-+ +-+--+-+ +-+--+-+ +-+--+-
+
|      | |      | |      | |      |   |      | |      | |      | |
|
|      | |      | |      | |      |   |      | |      | |      | |
|
+-+--+-+ +-+--+-+ +-+--+-+ +-+--+-+   +-+--+-+ +-+--+-+ +-+--+-+ +-+--+-
+
  |        |        |        |             |        |        |        |
  V        V        V        V             V        V        V        V
  1        2        3        4             5        6        7        8
```

fig.c)

d) Estimate the upper and lower bound for the number of the element switches in the n–universal scheme.

The 6–th competition — Chelyabinsk, 1972.

form	first day				second day		
8	159	160	161		166	167	168
9	162a	163	161	164	169	170	171
10	162b	163	165	164	166	172	173

159.
Given a rectangle ABCD, points M — the middle of [AD] side, N – the middle of [BC] side. Let us take a point P on the continuation of the [DC] segment over the point D. Let us denote the point of intersection of lines (PM) and (AC) as Q.
Prove that the angles QNM and MNP are equal.

160.
Given 50 segments on the line. Prove that one of the following statements is valid:

- 1. Some 8 segments have the common point.
- 2. Some 8 segments do not intersect each other.

161.
Find the maximal x such that the expression $4^{27} + 4^{1000} + 4^{x}$ is the exact square.

162.
a) Let a,n,m be natural numbers, a > 1. Prove that if $(a^m + 1)$ is divisible by $(a^n + 1)$ than m is divisible by n.
b) Let a,b,n,m be natural numbers, a>1, a and b are relatively prime. Prove that if (a^m+b^m) is divisible by (a^n+b^n) than m is divisible by n.

163.

```
            2
           / \
         /     \
        4       3
      /   \    /  \
     16    5   9    4
    / \   / \  /\   / \
```

The triangle table is built according to the rule: You put the natural number a>1 in the upper row, and then You write under the number k from the left side k^2, and from the right side — (k+1). For example, if a = 2, You get the table on the picture. Prove that all the numbers on each particular line are different.

164.
Given several squares with the total area 1. Prove that You can pose hem in the square of the area 2 without any intersections.

165.
Let O be the intersection point of the of the convex quadrangle ABCD iagonals. Prove that the line drawn through the points of ntersection of the medians of AOB and COD triangles is orthogonal to he line drawn through the points of intersection of the heights of BOC and AOD triangles.

166.
Each of the 9 straight lines divides the given square onto two quadrangles with the areas related as 2:3. Prove that there exist three of them intersecting in one point.

167.

The 7–angle $A_1A_2A_3A_4A_5A_6A_7$ is inscribed in a circle. Prove that if the centre of the circle is inside the 7–angle, than the sum of A_1,A_2 and A_3 angles is less than 450 degrees.

168.
A game for two. One gives a digit and the second substitutes it instead of a star in the following difference:

```
**** – **** =
```

Then the first gives the next digit, and so on 8 times. The first wants to obtain the greatest possible difference, the second — the least. Prove that:

- 1. The first can operate in such a way that the difference would be not less than 4000, not depending on the second's behaviour.
- 2. The second can operate in such a way that the difference would be not greater than 4000, not depending on the first's behaviour.

169.
Let x,y be positive numbers, s — the least of { x, (y+ 1/x), 1/y}.
What is the greatest possible value of s? To what x and y does it correspond?

170.
The point O inside the convex polygon makes isosceles triangle with all the pairs of its vertices.
Prove that O is the centre of the outscribed circle.

171.
Is it possible to put the numbers 0,1 or 2 in the unit squares of the cross–lined paper 100x100 in such a way, that every rectangle 3x4 (and 4x3) would contain three zeros, four ones and five twos?

172.
Let the sum of positive numbers x_1, x_2, ... , x_n be 1.
Let s be the greatest of the numbers $\{x_1/(1+x_1), x_2/(1+x_1+x_2), \dots x_n/(1+x_1+\dots+x_n)\}$.
What is the minimal possible s? What x_i correspond it.

/* Derivatives were not included in the school plans in Russia that time. They expected another solution. –VAP */

173.
One–round hockey tournament is finished (each plays with each one time, the winner gets 2 points, looser – – 0, and 1 point for draw). For arbitrary subgroup of teams there exists a team (may be from that subgroup) that has got an odd number of points in the games with the teams of the subgroup. Prove that there was even number of the participants.

The 7–th competition — Kishenew, 1973.

```
form        first day        second day
    8  174a  175   176  | 182   183   184ab
    9  174b  177   178  | 179   185   186   184c
   10  180   177   181  | 187   188   184c
```

174.
Fourteen coins are submitted to the judge. An expert knows, that the coins from number one to seven are false, and from 8 to 14 — normal. The judge is sure only that all the true coins have the same weight and all the false coins weights equal each other, but are less then the weight of the true coins. The expert has the scales without weights.

a) The expert wants to prove, that the coins 1—7 are false. How can he do it in three weighings?
b) How can he prove, that the coins 1—7 are false and the coins 8—14 are true in three weighings?

175.
Prove that 9–digit number, that contains all the decimal digits except zero and does not ends with 5 can not be exact square.

176.
Given n points, $n > 4$. Prove that You can connect them with arrows, in such a way, that You can reach every point from every other point, having passed through one or two arrows. (You can connect every pair with one arrow only, and move along the arrow in one direction only.)

177.
Given an angle with the vertex O and a circle touching its sides in the points A and B. A ray is drawn from the point A parallel to [OB). It intersects with the circumference in the point C. The segment [OC] intersects the circumference in the point E. The straight lines (AE) and (OB) intersect in the point K. Prove that $|OK| = |KB|$.

178.
The real numbers a,b,c satisfy the condition: for all x, such that for $-1 \le x \le 1$, the inequality

$$| ax^2 + bx + c | \le 1 \text{ is held.}$$

Prove that for the same x

$$| cx^2 + bx + a | \le 2.$$

179.
The tennis federation has assigned numbers to 1024 sportsmen, participating in the tournament, according to their skill. (The tennis federation uses the olympic system of tournaments. The looser in the pair leaves, the winner meets with the winner of another pair. Thus, in the second tour remains 512 participants, in the third — 256, et.c.
The winner is determined after the tenth tour.) It comes out, that in the play between the sportsmen whose numbers differ more than on 2 always win that whose number is less. What is the greatest possible number of the winner?

180.
The square polynomial $f(x)=ax^2+bx+c$ is of such a sort, that the equation $f(x)=x$ does not have real roots. Prove that the equation $f(f(x))$ does not have real roots also.

181.
n squares of the infinite cross–lined sheet of paper are painted with black colour (others are white). Every move all the squares of the sheet change their colour simultaneously. The square gets the colour, that had the majority of three ones: the square itself, its neighbour from the right side and its neighbour from the upper side.

a) Prove that after the finite number of the moves all the black squares will disappear.

b) Prove that it will happen not later than on the n–th move.

182.
Three similar acute–angled triangles AC_1B, BA_1C and CB_1A are built on the outer side of the acute–angled triangle ABC. (Equal triples of the angles are AB_1C, ABC_1, A_1BC and BA_1C, BAC_1, B_1AC.)
a) Prove that the circumferences outscribed around the outer triangles intersect in one point.
b) Prove that the straight lines AA_1, BB_1 and CC_1 intersect in the same point.

183.
N men are not acquainted each other. You need to introduce some of them to some of them in such a way, that all the men will have different number of the acquaintances. Prove that it is possible for all N.

184.
The king have revised the chess–board 8x8 having visited all the fields once only and returned to the starting point. When his trajectory was drawn (the centres of the squares were connected with the straight lines), a closed broken line without self–intersections appeared.
a) Give an example that the king could make 28 steps parallel the sides of the board only.
b) Prove that he could not make less than 28 such a steps.
c) What is the maximal and minimal length of the broken line if the side of a field is 1?

185.
Given a triangle with a,b,c sides and with the area 1. $a \ge b \ge c$.
Prove that $b^2 \ge 2$.

186.
Given a convex n–angle with pairwise (mutually) non-parallel sides

> /* who knows Russian — a letter "r" had been broken on the organising committee's typewriter – and it became an inexhaustible source of jokes for some years */

and a point inside it. Prove that there are not more than n straight lines coming through that point and halving the area of the n–angle.

187.
Prove that for every positive x_1, x_2, x_3, x_4, x_5 holds inequality:

$$(x_1 + x_2 + x_3 + x_4 + x_5)^2 \ge 4(x_1x_2 + x_3x_4 + x_5x_1 + x_2x_3 + x_4x_5).$$

188.
Given 4 points in three–dimensional space, not lying in one plane. What is the number of such a parallelepipeds (bricks), that each point is a vertex of each parallelepiped?

The 8–th competition — Erevan, 1974.

form	first day				second day			
8	189abc	190	191		197	198	199	200a
9	190	192	189d	193	201	202	200b	
10	194	195	196	193	203	204	200b	

189.
a,b,c) Given some cards with either "–1" or "+1" written on the opposite side. You are allowed to choose a triple of cards and ask about the product of the three numbers on the cards. What is the minimal number of questions allowing to determine all the numbers on the cards

a) for 30 cards,
b) for 31 cards,
c) for 32 cards.

(You should prove, that You cannot manage with less questions.)

d) Fifty abovementioned cards are lying along the circumference. You are allowed to ask about the product of three consecutive numbers only. You need to determine the product af all the 50 numbers. What is the minimal number of questions allowing to determine it?

190.
Among all the numbers representable as $36^k - 5^l$ (k and l are natural numbers) find the smallest.
Prove that it is really the smallest.

191.
a) Each of the side of the convex hexagon (6–angle) is longer than 1. Does it necessary have a diagonal longer than 2?
b) Each of the main diagonals of the convex hexagon is longer than 2. Does it necessary have a side longer than 1?

192.
Given two circles with the radiuses R and r, touching each other from the outer side. Consider all the trapezoids, such that its lateral sides touch both circles, and its bases touch different circles. Find the shortest possible lateral side.

193.
Given n vectors of unit length in the plane. The length of their total sum is less than one. Prove that You can rearrange them to provide the property: for every k, $k \le n$, the length of the sum of the first k vectors is less than 2.

194.
Find all the real a,b,c such that the equality

$$|ax + by + cz| + |bx + cy + az| + |cx + ay + bz| = |x| + |y| + |z|$$

is valid for all the real x,y,z.

195.
Given a square ABCD. Points P and Q are in the sides [AB] and [BC] respectively. |BP| = |BQ|. Let H be the base of the perpendicular from the point B to the segment [PC].
Prove that the angle DHQ is a right one.

196.
Given some red and blue points. Some of them are connected by the segments. Let us call "exclusive" the point, if its colour differs from the colour of more than half of the connected points. Every move one arbitrary "exclusive" point is repainted to the other colour. Prove that after the finite number of moves there will remain no "exclusive" points.

197.

Find all the natural n and k such that n^n has k digits and k^k has n digits.

198.

Given points D and E on the legs [CA] and [CB], respectively, of the isosceles right triangle. |CD| = |CE|. The extensions of the perpendiculars from D and C to the line AE cross the hypotenuse AB in the points K and L. Prove that |KL| = |LB|.

199.

Two are playing the game "cats and rats" on the chess–board 8x8. The first has one piece — a rat, the second — several pieces — cats. All the pieces have four available moves — up, down, left, right — to the neighbour field, but the rat can also escape from the board if it is on the boarder of the chess–board. If they appear on the same field – the rat is eaten. The players move in turn, but the second can move all the cats in independent directions.

a) Let there be two cats. The rat is on the interior field. Is it possible to put the cats on such a fields on the border that they will be able to catch the rat?
b) Let there be three cats, but the rat moves twice during the first turn. Prove that the rat can escape.

200.

a) Prove that You can rearrange the numbers 1, 2, ... , 32 in such a way, that for every couple of numbers none of the numbers between them will equal their arithmetic mean.
b) Can You rearrange the numbers 1, 2, ... , 100 in such a way, that for every couple of numbers none of the numbers between them will equal their arithmetic mean?

201.

Find all the three–digit numbers such that it equals to the arithmetic mean of the six numbers obtained by rearranging its digits.

202.

Given a convex polygon. You can put no triangle with area 1 inside it. Prove that You can put the polygon inside a triangle with the area 4.

203.

Given a function f(x) on the segment $0 \le x \le 1$. For all x, $f(x) \ge 0$; $f(1)=1$. For all the couples of $\{x_1,x_2\}$ such, that all the arguments are in the segment $f(x_1+x_2) \ge f(x_1)+f(x_2)$.

a) Prove that for all x holds $f(x) \le 2x$.
b) Is the inequality $f(x) \le 1.9x$ valid?

204.

Given a triangle ABC with the are 1. Let A',B' and C' are the middles of the sides [BC], [CA] and [AB] respectively. What is the minimal possible area of the common part of two triangles A'B'C' and KLM, if the points K,L and M are lying on the segments [AB'], [CA'] and [BC'] respectively?

The 9–th competition — Saratov, 1975.

form	first day				second day		
8	205a	206	207	208a	213	214	215
9	209	206	210	208b	216	215	217
10	211	212	205b	208	214	218	219

205.

a) The triangle ABC was turned around the centre of the outscribed circle by the angle less than 180 degrees and thus was obtained the triangle A_1B_1C_1. The corresponding segments [AB] and [A_1B_1] intersect in the point C_2; [BC] and [B_1C_1] — A_2; [AC] and [A_1C_1] — B_2. Prove that the triangle A_2B_2C_2 is similar to the triangle ABC.

b) The quadrangle ABCD was turned around the centre of the outscribed circle by the angle less than 180 degrees and thus was obtained the quadrangle A_1B_1C_1D_1. Prove that the points of intersection of the corresponding lines ((AB) and (A_1B_1), (BC) and (B_1C_1), (CD) and (C_1D_1), (DA) and (D_1A_1)) are the vertices of the parallelogram.

206.

Given a triangle ABC with the unit area. The first player chooses a point X on the side [AB], than the second — Y on [BC] side, and, finally, the first chooses a point Z on [AC] side. The first tries to obtain the greatest possible area of the XYZ triangle, the second — the smallest. What area can obtain the first for sure and how?

207.

What is the smallest perimeter of the convex 32–angle, having all the vertices in the nodes of cross–lined paper with the sides of its squares equal to 1?

208.

a) Given a big square consisting of 7x7 squares. You should mark the centres of k points in such a way, that no quadruple of the marked points will be the vertices of a rectangle with the sides parallel to the sides of the given squares. What is the greatest k such that the problem has solution?

b) The same problem for {13}x{13} square.

209.

Denote the middles of the convex hexagon $A_1A_2A_3A_4A_5A_6$ diagonals A_6A_2, A_1A_3, A_2A_4, A_3A_5, A_4A_6, A_5A_1 as B_1, B_2, B_3, B_4, B_5, B_6 respectively.

Prove that if the hexagon $B_1B_2B_3B_4B_5B_6$ is convex, than its area equals to the quarter of the initial hexagon.

210.

Prove that it is possible to find 2^{n+1} of 2^n digit numbers containing only "1" and "2" as digits, such that every two of them distinguish at least in 2^{n-1} digits.

211.

Given a finite set of polygons in the plane. Every two of them have a common point. Prove that there exists a straight line, that crosses all the polygons.

212.

Prove that for all the positive numbers a,b,c the following inequality is valid:

$$a^3+b^3+c^3+3abc>ab(a+b)+bc(b+c)+ac(a+c).$$

213.

Three flies are crawling along the perimeter of the ABC triangle in such a way, that the centre of their masses is a constant point. One of the flies has already passed along all the perimeter. Prove that the centre of the flies' masses coincides with the centre of masses of the ABC triangle. (The centre of masses for the triangle is the point of medians intersection.)

214.
Several zeros, ones and twos are written on the blackboard. An anonymous clean in turn pairs of different numbers, writing, instead of cleaned, the number not equal to each. (0 instead of pair {1,2}; 1 instead of {0,2}; 2 instead of {0,1}). Prove that if there remains one number only, it does not depend on the processing order.

215.
Given a horizontal strip on the plane (its sides are parallel lines) and n lines intersecting the strip. Every two of them intersect inside the strip, and not a triple has a common point. Consider all the paths along the segments of those lines, starting on the lower side of the strip and ending on the upper side with the properties: moving along such a path we are constantly rising up, and, having reached the intersection, we are obliged to turn to another line. Prove that:

a) there are not less than n/2 such a paths without common points;
b) there is a path consisting of not less than of n segments;
c) there is a path that goes along not more than along n/2+1 lines;
d) there is a path that goes along all the n lines.

216.
For what k is it possible to construct a cube kxkxk of the black and white cubes 1x1x1 in such a way that every small cube has the same colour, that have exactly two his neighbours. (Two cubes are neighbours, if they have the common face.)

217.
Given a polynomial P(x) with

a) natural coefficients;
b) integer coefficients;

Let us denote with a_n the sum of the digits of P(n) value.
Prove that there is a number encountered in the sequence $a_1, a_2, \dots, a_n, \dots$ infinite times.

218.
The world and the european champion are determined in the same tournament carried in one round. There are 20 teams and k of them are european. The european champion is determined according to the results of the games only between those k teams. What is the greatest k such that the situation, when the single european champion is the single world outsider, is possible if:

a) it is hockey (draws allowed)?
b) it is volleyball (no draws)?

219.

a) Given real numbers a_1,a_2,b_1,b_2 and positive p_1,p_2,q_1,q_2. Prove that in the table 2x2

$$(a_1 + b_1)/(p_1 + q_1)\ ,\ (a_1 + b_2)/(p_1 + q_2)$$
$$(a_2 + b_1)/(p_2 + q_1)\ ,\ (a_2 + b_2)/(p_2 + q_2)$$

there is a number in the table, that is not less than another number in the same row and is not greater than another number in the same column (a saddle point).

b) Given real numbers $a_1, a_2, \dots, a_n$; $b_1, b_2, \dots, b_n$ and positive $p_1, p_2, \dots, p_n$; $q_1, q_2, \dots, q_n$. We build the table nxn, with the numbers $(0 < i,j \le n)$

$$(a_i + b_j)/(p_i + q_j)$$

in the intersection of the i–th row and j–th column. Prove that there is a number in the table, that is not less than arbitrary number in the same row and is not greater than arbitrary number in the same column (a saddle point).

The 10–th competition — Dushanbe, 1976.

form	first day					second day		
8	220	221	222ab	223		229	230	231
9	222b	224	223	225		230	232	231
10	223	226	227	228	225	233	234	231

220.
There are 50 exact watches lying on a table. Prove that there exist a certain moment, when the sum of the distances from the centre of the table to the ends of the minute hands is more than the sum of the distances from the centre of the table to the centres of the watches.

221.
A row of 1000 numbers is written on the blackboard. We write a new row, below the first according to the rule: We write under every number a the natural number, indicating how many times the number a is encountered in the first line. Then we write down the third line: under every number b — the natural number, indicating how many times the number b is encountered in the second line, and so on.

a) Prove that there is a line that coincides with the preceding one.
b) Prove that the eleventh line coincides with the twelfth.
c) Give an example of the initial line such, that the tenth row differs from the eleventh.

222.
Given three circumferences of the same radius in a plane.

a) All three are crossing in one point K. Consider three arcs AK,CK,EK: the A,C,E are the points of the circumferences intersection and the arcs are taken in the clockwise direction. (Sorry, no picture. Every arc is inside one circle, outside the second and on the border of the third one) Prove that the sum of the arcs is 180 degrees.
b) Consider the case, when the three circles give a curvilinear triangle BDF as there intersection (instead of one point K). Prove that the sum of the AB, CD and EF arcs is 180 degrees. (The arcs are taken in the clockwise direction. Every arc is inside one circle, outside the second and on the border of the third one)

223.
The natural numbers x_1 and x_2 are less than 1000. We build a sequence:

$$x_3 = |x_1 - x_2|;$$
$$x_4 = \min \{ |x_1 - x_2|, |x_1 - x_3|, |x_2 - x_3|\};$$
$$.......$$
$$x_k = \min \{ |x_i - x_j|; 0 < i < j < k\};$$
$$.......$$

Prove that $x_{21} = 0$.

224.
Can You mark the cube's vertices with the three–digit binary numbers in such a way, that the numbers at all the possible couples of neighbouring vertices differ in at least two digits?

225.
Given 4 vectors a,b,c,d in the plane, such that a + b + c + d = 0.
Prove the following inequality:

$$|a| + |b| + |c| + |d| \geq |a + d| + |b + d| + |c + d|.$$

226.
Given right 1976–angle. The middles of all the sides and diagonals are marked. What is the greatest number of the marked points lying on one circumference?

227.

There are n rectangles drawn on the rectangular sheet of paper with the sides of the rectangles parallel to the sheet sides. The rectangles do not have pairwise common interior points. Prove that after cutting out the rectangles the sheet will split into not more than n+1 part.

228.

There are three straight roads. Three pedestrians are moving along those roads, and they are NOT on one line in the initial moment. Prove that they will be one line not more than twice.

229.

Given a chess–board 99x99 with a set F of fields marked on it (the set is different in three tasks). There is a beetle sitting on every field of the set F. Suddenly all the beetles have raised into the air and flied to another fields of the same set. The beetles from the neighbouring fields have landed either on the same field or on the neighbouring ones (may be far from their starting point). (We consider the fields to be neighbouring if they have at least one common vertex.). Consider a statement: "There is a beetle, that either stayed on the same field or moved to the neighbouring one". Is it always valid if the figure F is:

a) A central cross, i.e. the union of the 50–th row and the 50–th column?
b) A window frame, i.e. the union of: the 1–st, 50–th and 99–th rows and the 1–st, 50–th and 99–th columns?
c) All the chess–board?

230.

Let us call "big" a triangle with all sides longer than 1. Given a equilateral triangle with all the sides equal to 5. Prove that:

a) You can cut 100 big triangles out of given one.
b) You can divide the given triangle onto 100 big nonintersecting ones fully covering the initial one.
c) The same as b), but the triangles either do not have common points, or have one common side, or one common vertex.
d) The same as c), but the initial triangle has the side 3.

231.

Given natural n. We shall call "universal" such a sequence of natural number $a_1, a_2, \ldots, a_k$; $k \geq n$, if we can obtain every transposition of the first n natural numbers (i.e such a sequence of n numbers, that every one is encountered only once) by deleting some its members. (Examples: (1,2,3,1,2,1,3) is universal for n=3, and (1,2,3,2,1,3,1) — not, because You can't obtain (3,1,2) from it.) The goal is to estimate the length of the shortest universal sequence for given n.

a) Give an example of the universal sequence of n^2 members.
b) Give an example of the universal sequence of $(n^2 - n + 1)$ members.
c) Prove that every universal sequence contains not less than $n(n + 1)/2$ members
d) Prove that the shortest universal sequence for n=4 contains 12 members
e) Find as short universal sequence, as You can. The Organising Committee knows the method for $(n^2 - 2n + 4)$ members.

232.

n numbers are written down along the circumference. Their sum equals to zero, and one of them equals 1.

a) Prove that there are two neighbours with their difference not less than n/4.
b) Prove that there is a number that differs from the arithmetic mean of its two neighbours not less than on $8/\{n^2\}$. .
c) Try to improve the previous estimation, i.e what number can be used instead of 8?
d) Prove that for n=30 there is a number that differs from the arithmetic mean of its two neighbours not less than on 2/113; give an example of such 30 numbers along the circumference, that not a single number differs from the arithmetic mean of its two neighbours more than on 2/113.

233.

Given right n–angle wit the point O — its centre. All the vertices are marked either with +1 or –1. We may change all the signs in the vertices of right k–angle ($2 \le k \le n$) with the same centre O. (By 2–angle we understand a segment, being halved by O.)
Prove that in a), b) and c) cases there exists such a set of (+1)s and (–1)s, that we can never obtain a set of (+1)s only.

a) $n = 15$;
b) $n = 30$; c) $n > 2$;
c) Let us denote K(n) the maximal number of (+1) and (–1) sets such, that it is impossible to obtain one set from another. Prove, for example, that $K(200) = 2^{80}$.

234.
Given a sphere of unit radius with the big circle (i.e of unit radius) that will be called "equator". We shall use the words "pole", "parallel","meridian" as self–explanatory.

a) Let g(x), where x is a point on the sphere, be the distance from this point to the equator plane. Prove that g(x) has the property

   if x_1, x_2, x_3 are the ends of the pairwise orthogonal radiuses, than

$$g(x_1)^2 + g(x_2)^2 + g(x_3)^2 = 1. \quad (*)$$

Let function f(x) be an arbitrary nonnegative function on a sphere that satisfies (*) property.

b) Let x_1 and x_2 points be on the same meridian between the north pole and equator, and x_1 is closer to the pole than x_2. Prove that $f(x_1) > f(x_2)$.
c) Let y_1 be closer to the pole than y_2. Prove that $f(y_1) > f(y_2)$.
d) Let z_1 and z_2 be on the same parallel. Prove that $f(z_1) = f(z_2)$.
e) Prove that for all x $f(x) = g(x)$.

The 11–th competition — Tallinn, 1977.

form	first day					second day			
8	235	236	237b	238		243	244ab	245	246
9	237a	239	235	240		247	248	249	250
10	237a	239	241	242	235	251	244	246	

235.
Given a closed broken line without self–intersections in a plane. Not a triple of its vertices belongs to one straight line. Let us call "special" a couple of line's segments if the one's continuation intersects another.
Prove that there is even number of special pairs.

236.
Given several points, not all lying on one straight line. Some number is assigned to every point. It is known, that if a straight line contains two or more points, than the sum of the assigned to those points equals zero.
Prove that all the numbers equal to zero.

237.

a) Given a circle with two inscribed triangles T_1 and T_2. The vertices of T_1 are the middles of the arcs with the ends in the vertices of T_2. Consider a hexagon — the intersection of T_1 and T_2. Prove that its main diagonals are parallel to T_1 sides and are intersecting in one point.
b) The segment, that connects the middles of the arcs AB and AC of the circle outscribed around the ABC triangle, intersects [AB] and [AC] sides in D and K points. Prove that the points A,D,K and O — the centre of the circle — are the vertices of a diamond.

238.

Several black an white checkers (tokens?) are standing along the circumference. Two men remove checkers in turn. The first removes all the black ones that had at least one white neighbour, and the second — all the white ones that had at least one black neighbour. They stop when all the checkers are of the same colour.

a) Let there be 40 checkers initially. Is it possible that after two moves of each man there will remain only one (checker)?
b) Let there be 1000 checkers initially. What is the minimal possible number of moves to reach the position when there will remain only one (checker)?

239.
Given infinite sequence a_n. It is known that the limit of $b_n=a_{n+1}-a_n/2$ equals zero. Prove that the limit of a_n equals zero.

240.
There are direct routes from every city of a certain country to every other city. The prices are known in advance. Two tourists (they do not necessary start from one city) have decided to visit all the cities, using only direct travel lines. The first always chooses the cheapest ticket to the city, he has never been before (if there are several — he chooses arbitrary destination among the cheapests). The second — the most expensive (they do not return to the first city). Prove that the first will spend not more money for the tickets, than the second.

/* The fact seems to be evident, but the proof is not easy — VAP */

241.

Every vertex of a convex polyhedron belongs to three edges. It is possible to outscribe a circle around all its faces. Prove that the polyhedron can be inscribed in a sphere.

242.
The polynomial $x^{10} + ?x^9 + ?x^8 + ... + ?x + 1$ is written on the blackboard. Two players substitute (real) numbers instead of one of the question marks in turn. (9 turns total.) The first wins if the polynomial will have no real roots. Who wins?

243.
Seven elves are sitting at a round table. Each elf has a cup. Some cups are filled with some milk. Each elf in turn and clockwise divides all his milk between six other cups. After the seventh has done this, every cup was containing the initial amount of milk. How much milk did every cup contain, if there was three litres of milk total?

244.
Let us call "fine" the 2n–digit number if it is exact square itself and the two numbers represented by its first n digits (first digit may not be zero) and last n digits (first digit may be zero, but it may not be zero itself) are exact squares also.

a) Find all two– and four–digit fine numbers.
b) Is there any six–digit fine number?
c) Prove that there exists 20–digit fine number.
d) Prove that there exist at least ten 100–digit fine numbers.
e) Prove that there exists 30–digit fine number.

245.
Given a set of n positive numbers. For each its nonempty subset consider the sum of all the subset's numbers. Prove that You can divide those sums onto n groups in such a way, that the least sum in every group is not less than a half of the greatest sum in the same group.

246.

There are 1000 tickets with the numbers 000, 001, ... , 999; and 100 boxes with the numbers 00, 01, ... , 99. You may put a ticket in a box, if You can obtain the box number from the ticket number by deleting one digit. Prove that:

a) You can put all the tickets in 50 boxes;
b) 40 boxes is not enough for that;
c) it is impossible to use less than 50 boxes.
d) Consider 10000 4–digit tickets, and You are allowed to delete two digits. Prove that 34 boxes is enough for storing all the tickets.
e) What is the minimal used boxes set in the case of k–digit tickets?

247.
Given a square 100x100 on the sheet of cross–lined paper. There are several broken lines drawn inside the square. Their links consist of the small squares sides. They are neither pairwise– nor self–intersecting (have no common points). Their ends are on the big square boarder, and all the other vertices are in the big square interior.
Prove that there exists (in addition to four big square angles) a node (corresponding to the cross–lining family, inside the big square or on its side) that does not belong to any broken line.

248.
Given natural numbers $x_1,x_2,...,x_n;y_1,y_2,...,y_m$. The following condition is valid:

$$(x_1+x_2+...+x_n)=(y_1+y_2+...+y_m)<mn. \quad (*)$$

Prove that it is possible to delete some terms from (*) (not all and at least one) and to obtain another valid condition.

249.
Given 1000 squares on the plane with their sides parallel to the coordinate axes. Let M be the set of those squares centres. Prove that You can mark some squares in such a way, that every point of M will be contained not less than in one and not more than in four marked squares.

250.
Given scales and a set of n different weights. We take weights in turn and add them on one of the scales sides. Let us denote "L" the scales state with the left side down, and "R" — with the right side down.

a) Prove that You can arrange the weights in such an order, that we shall obtain the sequence LRLRLRLR... of the scales states. (That means that the state of the scales will be changed after putting every new weight.)
b) Prove that for every n–letter word containing R's and L's only You can arrange the weights in such an order, that the sequence of the scales states will be described by that word.

251.
Let us consider one variable polynomials with the senior coefficient equal to one. We shall say that two polynomials P(x) and Q(x) commute, if P(Q(x))=Q(P(x)) (i.e. we obtain the same polynomial, having collected the similar terms).

a) For every a find all Q such that the Q degree is not greater than three, and Q commutes with $(x^2 – a)$.
b) Let P be a square polynomial, and k is a natural number. Prove that there is not more than one commuting with P k–degree polynomial.
c) Find the 4–degree and 8–degree polynomials commuting with the given square polynomial P.
d) R and Q commute with the same square polynomial P. Prove that Q and R commute.
e) Prove that there exists a sequence $P_2, P_3, ... , P_n, ...$ (P_k is k–degree polynomial), such that $P_2(x) = x^2 – 2$, and all the polynomials in this infinite sequence pairwise commute.

The 12–th competition — Tashkent, 1978.

`form        first day                        second day`

8	252	253	254	255ab	260	261	262	263
9	252	253	256	257	260	261	264	265
10	258	259	255cde	257	260	266	267	268

252.
Let a_n be the closest to $\sqrt{t(n)}$ integer. Find the sum $1/a_1 + 1/a_2 + ... + 1/a_{1980}$.

253.
Given a quadrangle ABCD and a point M inside it such that ABMD is a parallelogram. the angle CBM equals to CDM.
Prove that the angle ACD equals to BCM.

254.
Prove that there is no m such that $(1978^m – 1)$ is divisible by $(1000^m – 1)$.

255.
Given a finite set K_0 of points (in the plane or space). The sequence of sets K_1, K_2, ... , K_n, ... is build according to the rule: we take all the points of K_i, add all the symmetric points with respect to all its points, and, thus obtain K_{i+1}.

a) Let K_0 consist of two points A and B with the distance 1 unit between them. For what n the set K_n contains the point that is 1000 units far from A?
b) Let K_0 consist of three points that are the vertices of the equilateral triangle with the unit square. Find the area of minimal convex polygon containing K_n. K_0 below is the set of the unit volume tetrahedron vertices.
c) How many faces contain the minimal convex polyhedron containing K_1?
d) What is the volume of the abovementioned polyhedron?
e) What is the volume of the minimal convex polyhedron containing K_n?

256.
Given two heaps of checkers. the bigger contains m checkers, the smaller — n (m>n). Two players are taking checkers in turn from the arbitrary heap. The players are allowed to take from the heap a number of checkers (not zero) divisible by the number of checkers in another heap. The player that takes the last checker in any heap wins.

a) Prove that if m > 2n, than the first can always win.
b) Find all x such that if m > xn, than the first can always win.

257.
Prove that there exists such an infinite sequence $\{x_i\}$, that for all m and all k (m<>k) holds the inequality $|x_m – x_k| > 1/|m–k|$.

258.
Let $f(x) = x^2 + x + 1$. Prove that for every natural m>1 the numbers m, f(m), f(f(m)), ... are relatively prime.

259.
Prove that there exists such a number A that You can inscribe 1978 different size squares in the plot of the function y = A sin(x). (The square is inscribed if all its vertices belong to the plot.)

260.
Given three automates that deal with the cards with the pairs of natural numbers. The first, having got the card with (a,b), produces new card with (a+1,b+1); the second, having got the card with (a,b), produces new card with (a/2,b/2), if both a and b are even and nothing in the opposite case; the third, having got the pair of cards with (a,b) and (b,c) produces new card with (a,c). All the automates return the initial cards also. Suppose there was (5,19) card initially. Is it possible to obtain

a) (1,50)?
b) (1,100)?

c) Suppose there was (a,b) card initially $(a<b)$. We want to obtain (1,n) card. For what n is it possible?

261.
Given a circle with radius R and inscribed n–angle with area S. We mark one point on every side of the given polygon. Prove that the perimeter of the polygon with the vertices in the marked points is not less than 2S/R.

262.
The checker is standing on the corner field of a nxn chess–board. Each of two players moves it in turn to the neighbour (i.e. that has the common side) field. It is forbidden to move to the field, the checker has already visited. That who cannot make a move losts. a) Prove that for even n the first can always win, and if n is odd, than the second can always win.
c) Who wins if the checker stands initially on the neighbour to the corner field?

263.
Given n nonintersecting segments in the plane. Not a pair of those belong to the same straight line. We want to add several segments, connecting the ends of given ones, to obtain one nonselfintersecting broken line. Is it always possible?

264.
Given $0 < a \le x_1 \le x_2 \le ... \le x_n \le b$.
Prove that

$$(x_1+x_2+...+x_n)(1/x_1+1/x_2+...+1/x_n)\le((a+b)^2/4ab)n^2.$$

265.
Given a simple number p>3. Consider the set M of the pairs (x,y) with the integer coordinates in the plane such that $0 \le x < p$; $0 \le y < p$. Prove that it is possible to mark p points of M such that not a triple of marked points will belong to one line and there will be no parallelogram with the vertices in the marked points.

266.
Prove that for every tetrahedron there exist two planes such that the projection areas on those planes relation is not less than √t(2).

267.
Given $a_1, a_2, ... , a_n$. Define $b_k = (a_1 + a_2 + ... + a_k)/k$ for $1 \le k \le n$. Let

$$C = (a_1 - b_1)^2 + (a_2 - b_2)^2 + ... + (a_n - b_n)^2;$$
$$D = (a_1 - b_n)^2 + (a_2 - b_n)^2 + ... + (a_n - b_n)^2.$$

Prove that C ≤ <D ≤ 2C.

268.
Consider a sequence $x_n=(1+\surd t(2)+\surd t(3))^n$. Each member can be represented as

$$x_n=q_n+r_n\surd t(2)+s_n\surd t(3)+t_n\surd t(6),$$

where q_n, r_n, s_n, t_n are integers.
Find the limits of the fractions r_n/q_n, s_n/q_n, t_n/q_n.

The 13–th competition — Tbilisi, 1979.

form	first day			second day			
8	269	270	271	274	275	276	277
9	269	272	271	278	279	280	281
10	273	272	271	276	275	282	283

269.
What is the least possible relation of two isosceles triangles areas, if three vertices of the first one belong to three different sides of the second one?

270.
A grasshopper is hopping in the angle $x\geq 0$, $y\geq 0$ of the coordinate plane (that means that it cannot land in the point with negative coordinate). If it is in the point (x,y), it can either jump to the point $(x+1,y-1)$, or to the point $(x-5,y+7)$. Draw a set of such an initial points (x,y), that having started from there, a grasshopper cannot reach any point farther than 1000 from the point (0,0). Find its area.

271.
Every member of a certain parliament has not more than 3 enemies. Prove that it is possible to divide it onto two subparliaments so, that everyone will have not more than one enemy in his subparliament. (A is the enemy of B if and only if B is the enemy of A.)

272.
Some numbers are written in the notebook. We can add to that list the arithmetic mean of some of them, if it doesn't equal to the number, already having been included in it. Let us start with two numbers, 0 and 1. Prove that it is possible to obtain
a) 1/5;
b) an arbitrary rational number between 0 and 1.

273.
For every n the decreasing sequence $\{x_k\}$ satisfies a condition
$$x_1+x_4/2+x_9/3+...+x_{n^2}/n\leq 1.$$
Prove that for every n it also satisfies
$$x_1+x_2/2+x_3/3+...+x_n/n\leq 3.$$

274.
Given some points in the plane. For some pairs A,B the vector AB is chosen. For every point the number of the chosen vectors starting in that point equal to the number of the chosen vectors ending in that point. Prove that the sum of the chosen vectors equals to zero vector.

275.
What is the least possible number of the checkers being required
a) for the 8x8 chess–board;
b) for the nxn chess–board; to provide the property: Every line (of the chess–board fields) parallel to the side or diagonal is occupied by at least one checker?

276.
Find x and y (a and b parameters):
$$(x-y*\surd t(x^2-y^2))/(\surd t(1-x^2+y^2)) = a;$$
$$(y-x*\surd t(x^2-y^2))/(\surd t(1-x^2+y^2)) = b.$$

277.
Given some square carpets with the total area 4. Prove that they can fully cover the unit square.

278.
Prove that for the arbitrary numbers $x_1, x_2, \dots, x_n$ from the [0,1] segment
$$(x_1 + x_2 + ... + x_n + 1)^2 \geq 4(x_1^2 + x_2^2 + ... + x_n^2).$$

279.

Natural p and q are relatively prime. The [0,1] is divided onto (p+q) equal segments. Prove that every segment except two marginal contain exactly one from the (p+q–2) numbers {1/p, 2/p, ... , (p–1)/p, 1/q, 2/q, ... , (q–1q)}.

280.
Given the point O in the space and 1979 straight lines $l_1, l_2, \dots, l_{1979}$ containing it. Not a pair of lines is orthogonal. Given a point A_1 on l_1 that doesn't coincide with O. Prove that it is possible to choose the points A_i on l_i (i = 2, 3, ... , 1979) in so that 1979 pairs will be orthogonal:

A_1A_3 and l_2;
A_2A_4 and l_3;
....
$A_{i-1}A_{i+1}$ and l_i;
....
$A_{1977}A_{1979}$ and l_{1978};
$A_{1978}A_1$ and l_{1979};
$A_{1979}A_2$ and l_1

281.
The finite sequence $a_1, a_2, \dots, a_n$ of ones and zeroes should satisfy a condition:

for every k from 0 to (n–1) the sum

$$a_1a_{k+1} + a_2a_{k+2} + \dots + a_{n-k}a_n$$

should be odd.

a) Build such a sequence for n=25.
b) Prove that there exists such a sequence for some n > 1000.

282.
The convex quadrangle is divided by its diagonals onto four triangles. The circles inscribed in those triangles are equal. Prove that the given quadrangle is a diamond.

283.
Given n points (in sequence) $A_1, A_2, \dots, A_n$ on a line. All the segments A_1A_2, A_2,A_3, ... $A_{n-1}A_n$ are shorter than 1. We need to mark (k–1) points so that the difference of every two segments, with the ends in the marked points, is shorter than 1. Prove that it is possible

a) for k=3;
b) for every k less than (n–1).

The 14–th competition — Saratov, 1980.

form	first day				second day			
8	284	285	286	287	293	294	295	296
9	288	289	286	290	295	297	298	299
10	291	289	292	290	300	301	302	303

284.
All the two–digit numbers from 19 to 80 are written in a line without spaces. Is the obtained number (192021....7980) divisible by 1980?

285.
The vertical side of a square is divided onto n segments. The sum of the segments with even numbers lengths equals to the sum of the segments with odd numbers lengths. (n–1) lines parallel to the horizontal

sides are drawn from the segments ends, and, thus, n strips are obtained. The diagonal is drawn from the lower left corner to the upper right one. This diagonal divides every strip onto left and right parts.
Prove that the sum of the left parts of odd strips areas equals to the sum of the right parts of even strips areas.

286.
The load for the space station "Salute" is packed in containers. There are more than 35 containers, and the total weight is 18 metric tons. There are 7 one–way transport spaceships "Progress", each able to bring 3 metric tons to the station. It is known that they are able to take an arbitrary subset of 35 containers.
Prove that they are able to take all the load.

287.
The points M and P are the middles of [BC] and [CD] sides of a convex quadrangle ABCD. It is known that $|AM| + |AP| = a$. Prove that the ABCD area is less than $\{a^2\}/2$.

288.
Are there three simple numbers x,y,z, such that $x^2 + y^3 = z^4$?

289.
Given a point E on the diameter AC of the certain circle. Draw a chord BD to maximise the area of the quadrangle ABCD.

290.
There are several settlements on the bank of the Big Round Lake. Some of them are connected with the regular direct ship lines. Two settlements are connected if and only if two next (counterclockwise) to each ones are not connected.
Prove that You can move from the arbitrary settlement to another arbitrary settlement, having used not more than three ships.

291.
The six–digit decimal number contains six different non-zero digits and is divisible by 37. Prove that having transposed its digits You can obtain at least 23 more numbers divisible by 37.

292.
Find real solutions of the system

$$\sin x + 2 \sin(x+y+z) = 0,$$
$$\sin y + 3 \sin(x+y+z) = 0,$$
$$\sin z + 4 \sin(x+y+z) = 0.$$

293.
Given 1980 vectors in the plane, and there are some non-collinear among them. The sum of every 1979 vectors is collinear to the vector not included in that sum. Prove that the sum of all vectors equals to the zero vector.

294.
Let us denote with S(n) the sum of all the digits of n.
a) Is there such an n that $n+S(n)=1980$?
b) Prove that at least one of two arbitrary successive natural numbers is representable as $n + S(n)$ for some third number n.

295.
Some squares of the infinite sheet of cross–lined paper are red. Each 2x3 rectangle (of 6 squares) contains exactly two red squares. How many red squares can be in the 9x11 rectangle of 99 squares?

296.

An epidemic influenza broke out in the elves city. First day some of them were infected by the external source of infection and nobody later was infected by the external source. The elf is infected when visiting his ill friend. In spite of the situation every healthy elf visits all his ill friends every day. The elf is ill one day exactly, and has the immunity at least on the next day. There is no graftings in the city.
Prove that

a) If there were some elves immunised by the external source on the first day, the epidemic influenza can continue arbitrary long time.
b) If nobody had the immunity on the first day, the epidemic influenza will stop some day.

297.
Let us denote with P(n) the product of all the digits of n. Consider the sequence $n_{k+1} = n_k + P(n_k)$. Can it be unbounded for some n_1?

298.
Given equilateral triangle ABC. Some line, parallel to [AC] crosses [AB] and [BC] in M and P points respectively. Let D be the centre of PMB triangle, E – the middle of the [AP] segment.
Find the angles of DEC triangle.

299.
Let the edges of rectangular parallelepiped be x,y and z ($x<y<z$). Let $p=4(x+y+z)$, $s=2(xy+yz+zx)$ and $d=\surd t(x^2+y^2+z^2)$ be its perimeter, surface area and diagonal length, respectively.
Prove that

$$x < 1/3(p/4 - \surd t(d^2 - s/2)),$$
$$z > 1/3(p/4 + \surd t(d^2 - s/2)).$$

300.
The A set consists of integers only. Its minimal element is 1 and its maximal element is 100. Every element of A except 1 equals to the sum of two (may be equal) numbers being contained in A.
What is the least possible number of elements in A?

301.
Prove that there is an infinite number of such numbers B that the equation $[x^{3/2}] + [y^{3/2}] = B$ has at least 1980 integer solutions (x,y). ([z] denotes the greatest integer not exceeding z.)

302.
The edge [AC] of the tetrahedron ABCD is orthogonal to [BC], and [AD] is orthogonal to [BD]. Prove that the cosine of the angle between (AC) and (BD) lines is less than |CD|/|AB|.

303.
The number x from [0,1] is written as an infinite decimal fraction. Having rearranged its first five digits after the point we can obtain another fraction that corresponds to the number x_1. Having rearranged five digits of x_k from (k+1)–th till (k+5)–th after the point we obtain the number x_{k+1}.

a) Prove that the sequence x_i has limit.
b) Can this limit be irrational if we have started with the rational number?
c) Invent such a number, that always produces irrational numbers, no matter what digits were transposed.

The 15–th competition — Alma–Ata, 1981.

form	first day				second day			
8	304	305	306	307	315	316	317	318
9	308	309	310	311	319	320	321	322
10	311	312	312	314	323	324	325	326

304.
Two equal chess–boards (8x8) have the same centre, but one is rotated by 45 degrees with respect to another. Find the total area of black fields intersection, if the fields have unit length sides.

305.
Given points A,B,M,N on the circumference. Two chords $[MA_1]$ and $[MA_2]$ are orthogonal to (NA) and (NB) lines respectively.
Prove that (AA_1) and (BB_1) lines are parallel.

306.
Let us say, that a natural number has the property P(k) if it can be represented as a product of k succeeding natural numbers greater than 1.
a) Find k such that there exists n which has properties P(k) and P(k+2) simultaneously.
b) Prove that there is no number having properties P(2) and P(4) simultaneously.

307.
The rectangular table has four rows. The first one contains arbitrary natural numbers (some of them may be equal). The consecutive lines are filled according to the rule: we look through the previous row from left to the certain number n and write the number k if n was met k times. Prove that the second row coincides with the fourth one.

308.
Given real a. Find the least possible area of the rectangle with the sides parallel to the coordinate axes and containing the figure determined by the system of inequalities

$$y \le -x^2,$$
$$y \ge x^2 - 2x + a.$$

/* as there is no derivatives in the school program of the 9–th form, the participants had to use parabola properties –VAP */

309.

Three equilateral triangles ABC, CDE, EHK (the vertices are mentioned counterclockwise) are lying in the plane so, that the vectors [AD[and [DK[are equal.
Prove that the triangle BHD is also equilateral.

310.
There are 1000 inhabitants in a settlement. Every evening every inhabitant tells all his friends all the news he had heard the previous day. Every news becomes finally known to every inhabitant.
Prove that it is possible to choose 90 of inhabitants so, that if You tell them a news simultaneously, it will be known to everybody in 10 days.

311.
It is known about real a and b that the inequality $a \cos(x) + b \cos(3x) > 1$ has no real solutions.
Prove that $|b| \le 1$.

312.
The points K and M are the centres of the AB and CD sides of the convex quadrangle ABCD. The points L and M belong to two other sides and KLMN is a rectangle.
Prove that KLMN area is a half of ABCD area.

313.
Find all the sequences of natural k_n with two properties:

- for all n $k_n \le n \sqrt{t}(n)$;
- for all m>n $(k_n - k_m)$ is divisible by (m–n).

314.
Is it possible to fill a rectangular table with black and white squares (only) so, that the number of black squares will equal to the number of white squares, and each row and each column will have more than 75% squares of the same colour?

315.
The quadrangles AMBE, AHBT, BKXM, and CKXP are parallelograms. Prove that the quadrangle ABTE is also parallelogram. (the vertices are mentioned counterclockwise)

316.
Find the natural solutions of the equation $x^3 – y^3 = xy + 61$.

317.
Eighteen soccer teams have played 8 tours of a one–round tournament.
Prove that there is a triple of teams, having not met each other yet.

318.
The points C_1, A_1, B_1 belong to [AB], [BC], [CA] sides, respectively, of the ABC triangle.

$$|AC_1|/|C_1B| = |BA_1|/|A_1C| = |CB_1|/|B_1A| = 1/3.$$

Prove that the perimeter P of the ABC triangle and the perimeter p of the $A_1B_1C_1$ triangle satisfy inequality

$$P/2 < p < 3P/4.$$

319.
Positive numbers x,y satisfy equality $x^3+y^3=x–y$.
Prove that $x^2+y^2<1$.

320.
A pupil has tried to make a copy of a convex polygon, drawn inside the unit circle. He draw one side, from its end — another, and so on. Having finished, he has noticed that the first and the last vertices do not coincide, but are situated d units of length far from each other. The pupil draw angles precisely, but made relative error less than p in the lengths of sides.
Prove that $d < 4p$.

321.
A number is written in the each vertex of a cube. It is allowed to add one to two numbers written in the ends of one edge.
Is it possible to obtain the cube with all equal numbers if the numbers were initially as on the pictures:

```
  0-------0       7-------4       0-------1
 /|      /|      /|      /|      /|      /|
0-------0 |     6-------5 |     1-------0 |
| |     | |     | |     | |     | |     | |
| 0-----|-1     | 2-----|-3     | 0-----|-0
|/      |/      |/      |/      |/      |/
0-------0       1-------4       0-------0
```

fig.a fig.b fig.c

322.
Find n such that each of the numbers n,(n+1),...,(n+20) has the common divider greater than one with the number 30030 = 2*3*5*7*11*13.

323.
The natural numbers from 100 to 999 are written on separate cards. They are gathered in one pile with their numbers down in arbitrary order. Let us open them in sequence and divide into 10 piles according to the least significant digit. The first pile will contain cards with 0 at the end, ... , the tenth — with 9. Then we

shall gather 10 piles in one pile, the first — down, then the second, ... and the tenth — up. Let us repeat the procedure twice more, but the next time we shall divide cards according to the second digit, and the last time — to the most significant one.
What will be the order of the cards in the obtained pile?

324.
Six points are marked inside the 3x4 rectangle.
Prove that there is a pair of marked points with the distance between them not greater than √t(5).

325.
a) Find the minimal value of the polynomial $P(x,y) = 4 + (x^2)(y^4) + (x^4)(y^2) – 3(x^2)(y^2)$

/*the usage of derivatives is forbidden, as they were not included in the school programs—VAP*/

b) Prove that it cannot be represented as a sum of the squares of some polynomials of x,y.

326.
The segments [AD], [BE] and [CF] are the side edges of the _right_ triangle prism. (I mean, that the equilateral triangle is a base — VAP). Find all the points in its base ABC, situated on the equal distances from the (AE), (BF) and (CD) lines.

The 16–th competition — Odessa, 1982.

form	first day					second day			
8	327	328	329a	330	|	337	338	339	340
9	331	332	333	334	|	341	342	343	344
10	335	332	329b	336	|	345	346	347	348

327.
Given two points M and K on the circumference with radius r_1 and centre O_1. The circumference with radius r_2 and centre O_2 is inscribed in MO_1K angle. Find the MO_1KO_2 quadrangle area.

328.
Every member, starting from the third one, of two sequences $\{a_n\}$ and $\{b_n\}$ equals to the sum of two preceding ones. First members are: $a_1 = 1$, $a_2 = 2$, $b_1 = 2$, $b_2 = 1$. How many natural numbers are encountered in both sequences (may be on the different places)?

329.
a) Let m and n be natural numbers. For some nonnegative integers $k_1, k_2, ... , k_n$ the number $2^{k_1}+2^{k_2}+...+2^{k_n}$ is divisible by $(2^m–1)$. Prove that $n \geq m$.
b) Can You find a number, divisible by 111...1 (m times "1"), that has the sum of its digits less than m?

330.
A nonnegative real number is written at every cube's vertex. The sum of those numbers equals to 1. Two players choose in turn faces of the cube, but they cannot choose the face parallel to already chosen one (the first moves twice, the second — once).
Prove that the first player can provide the number, at the common for three chosen faces vertex, to be not greater than 1/6.

331.
Once upon a time, three boys visited a library for the first time. The first decided to visit the library every second day. The second decided to visit the library every third day. The third decided to visit the library every fourth day. The librarian noticed, that the library doesn't work on Wednesdays.

The boys decided to visit library on Thursdays, if they have to do it on Wednesdays, but to restart the day counting in these cases. They strictly obeyed these rules. Some Monday later I met them all in that library. What day of week was when they visited a library for the first time?

332.
The parallelogram ABCD isn't a diamond. The relation of the diagonal lengths |AC|/|BD| equals to k. The [AM) ray is symmetric to the [AD) ray with respect to the (AC) line. The [BM) ray is symmetric to the [BC) ray with respect to the (BD) line. (M point is those rays intersection.)
Find the |AM|/|BM| relation.

333.
3k points are marked on the circumference. They divide it onto 3k arcs. Some k of them have length 1, other k of them have length 2, the rest k of them have length 3. Prove that some two of the marked points are the ends of one diameter.

334.
Given a point M inside a right tetrahedron. Prove that at least one tetrahedron edge is seen from the M in an angle, that has a cosine not greater than –1/3. /* I mean that if A and B are the vertices, corresponding to that edge, $\cos(AMB) \leq -1/3$ */

335.
Three numbers a,b,c belong to $]0,\pi/2[$ interval. $\cos(a) = a$; $\sin(\cos(b)) = b$; $\cos(\sin(c)) = c$.
Sort those numbers in increasing order.

336.
The closed broken line M has odd number of vertices — $A_1,A_2,..., A_{2n+1}$ in sequence. Let us denote with S(M) a new closed broken line with vertices $B_1,B_2,...,B_{2n+1}$ — the middles of the first line links: B_1 is the middle of $[A_1A_2]$, ... , B_{2n+1} — of $[A_{2n+1}A_1]$.
Prove that in a sequence $M_1=S(M)$, ... , $M_k = S(M_{k-1})$, ... there is a broken line, homothetic to the M.

337.
All the natural numbers from 1 to 1982 are gathered in an array in an arbitrary order in computer's memory. The program looks through all the sequent pairs (first and second, second and third,...) and exchanges numbers in the pair, if the number on the lower place is greater than another. Then the program repeats the process, but moves from another end of the array. The number, that stand initially on the 100–th place reserved its place. Find that number.

338.
Cucumber river in the Flower city /* authors mean the N.Nosov's tale about the sort of elves country */ has parallel banks with the distance between them 1 metre. It has some islands with the total perimeter 8 metres. Mr. Know–All claims that it is possible to cross the river in a boat from the arbitrary point, and the trajectory will not exceed 3 metres. Is he right?

339.
There is a parabola $y = x^2$ drawn on the coordinate plane. The axes are deleted. Can You restore them with the help of compass and ruler?

340.
The square table nxn is filled by integers. If the fields have common side, the difference of numbers in them doesn't exceed 1.
Prove that some number is encountered not less than
a) not less than [n/2] times ([ ] mean the whole part);
b) not less than n times.

341.

Prove that the following inequality is valid for the positive x:

$$2^{(x^{1/12})} + 2^{(x^{1/4})} \ge 2^{(1 + x^{1/6})}.$$

342.
What minimal number of numbers from the set {1,2,...,1982} should be deleted to provide the property: none of the remained numbers equals to the product of two other remained numbers?

> /* This particular problem sounds bad due to the translator, but I had read the Moscow competition organising committee announcement with the word "prize" 7 times repeated in one sentence — VAP */

343.

Every square on the infinite sheet of cross–lined paper contains some real number.
Prove that some square contains a number that does not exceed at least four of eight neighbouring numbers.

344.
Given a sequence of real numbers $a_1, a_2, \dots, a_n$. Prove that it is possible to choose some of the numbers providing 3 conditions:

- a) not a triple of successive members is chosen;
- b) at least one of every triple of successive members is chosen;
- c) the absolute value of chosen numbers sum is not less that one sixth part of the initial numbers' absolute values sum.

345.
Given the square table nxn with (n–1) marked fields. Prove that it is possible to move all the marked fields below the diagonal by moving rows and columns.

346.
Prove that the following inequality holds for all real a and natural n:

$$|a|*|a-1|*|a-2|*\dots*|a-n| \ge F(a)*n!/(2^n),$$

F(a) is the distance from a to the closest integer.

347.
Can You find three polynomials P,Q,R of three variables x,y,z, providing the condition:

a)

$$P(x-y+z)^3 + Q(y-z-1)^3 + R(z-2x+1)^3 = 1;$$

b)

$$P(x-y+z)^3 + Q(y-z-1)^3 + R(z-x+1)^3 = 1;$$

for all x,y,z?

348.
The KLMN tetrahedron (triangle pyramid) vertices are situated inside or on the faces or on the edges of the ABCD tetrahedron.
Prove that KLMN perimeter is less than 4/3 ABCD perimeter.

The 17–th competition — Kishenew, 1983.

form	first day				second day			
8	349	350	351	352	360	361	362	363
9	353	354	355	356	364	365	366	367
10	357	354	358	359	360	368	369	370

349.

```
+---+---+---+---+
|   |   |   |   |
+---+---+---+---+
|   |   |   |   |
+---+---+---+---+
|   |   |   |   |
+---+---+---+---+
|   |   |   |   |
+---+---+---+---+
```

Every cell of the net, drawn on the picture has 1x1 size. Is it possible to represent this net as a union of the following sets:

a) Eight broken lines of length five each?
b) Five broken lines of length eight each?

350.

Three numbers were written with a chalk on the blackboard. The following operation was repeated several times: One of the numbers was cleared and the sum of two other numbers, decreased by 1, was written instead of it. The final set of numbers is {17, 1967, 1983} /* In some organisers opinion "17" is the most distinguishing number, and those three numbers were met on the picturesque jubilee slogans. In fact, in 1967 the competition was just renamed –VAP */. Is it possible to admit that the initial numbers were

a) {2, 2, 2}?
b) {3, 3, 3}?

351.

Three disks touch pairwise from outside in the points X,Y,Z. Then the radiuses of the disks were expanded by 2/√t(3) times, and the centres were reserved.
Prove that the XYZ triangle is completely covered by the expanded disks.

353.

Find all the solutions of the system

$$y^2 = x^3 - 3x^2 + 2x,$$
$$x^2 = y^3 - 3y^2 + 2y.$$

354.

Natural number k has n digits in its decimal notation. It was rounded up to tens, then the obtained number was rounded up to hundreds, and so on (n–1) times. Prove that The obtained number m satisfies inequality m < 18*k/13.
(Examples of rounding: 191–>190–>200; 135–>140–>100.)

355.

The point D is the middle of the [AB] side of the ABC triangle. The points E and F belong to [AC] and [BC] respectively. Prove that the DEF triangle area does not exceed the sum of the ADE and BDF triangles areas.

356.

The sequences a_n and b_n members are the last digits of $[(\sqrt{}t(10))^n]$ and $[(\sqrt{}t(2))^n]$ respectively (here [ ] denotes the whole part of a number). Are those sequences periodical?

357.

Two acute angles a and b satisfy condition

$$(\sin(a))^2 + (\sin(b))^2 = \sin(a+b).$$

Prove that a + b = π/2.

358.
The points A_1,B_1,C_1,D_1 and A_2,B_2,C_2,D_2 are orthogonal projections of the ABCD tetrahedron vertices on two planes.
Prove that it is possible to move one of the planes to provide the parallelness of (A_1A_2), (B_1B_2), (C_1C_2) and (D_1D_2) lines.

359.
The pupil is training in the square equation solution. Having the recurrent equation solved, he stops, if it doesn't have two roots, or solves the next equation, with the free coefficient equal to the greatest root, the coefficient at x equal to the least root, and the coefficient at x^2 equal to 1.
Prove that the process cannot be infinite. What maximal number of the equations he will have to solve?

360.
Given natural n,m,k. It is known that m^n is divisible by n^m; and n^k is divisible by k^n.
Prove that m^k is divisible by k^m.

361.
The Abba tribe language alphabet contains two letters only. Not a word of this language is a beginning of another word.
Can this tribe vocabulary contain 3 four–letter, 10 five–letter, 30 six–letter and 5 seven–letter words?

362.
Can You fill the squares of the infinite cross–lined paper with integers so, that the sum of the numbers in every 4x6 fields rectangle would be
a) 10?
b) 1?

363.
The points A_1,B_1,C_1 belong to [BC],[CA],[AB] sides of the ABC triangle respectively. The [AA_1], [BB_1], [CC_1] segments split the ABC onto 4 smaller triangles and 3 quadrangles. It is known, that the smaller triangles have the same area.
Prove that the quadrangles have equal areas. What is the quadrangle area, it the small triangle has the unit area?

364.
The kindergarten group is standing in the column of pairs. The number of boys equals the number of girls in each of the two columns. The number of mixed (boy and girl) pairs equals to the number of the rest pairs.
Prove that the total number of children in the group is divisible by eight.

365.
One side of the rectangle is 1cm. It is known that the rectangle can be divided by two orthogonal lines onto four rectangles, and each of the smaller rectangles has the area not less than 1 square centimetre, and one of them is not less than 2 square centimetres. What is the least possible length of another side of big rectangle?

366.
Given a point O inside ABC triangle. Prove that S_A*[OA) + S_B*[OB) + S_C*[OC) = [OO), where S_A, S_B, S_C denote BOC, COA, AOB triangles areas respectively ([AB) denotes vector from A to B , [OO) — zero vector).

367.
Given (2m+1) different integers, each absolute value is not greater than (2m–1).
Prove that it is possible to choose three numbers among them, with their sum equal to zero.

368.

The points D,E,F belong to the sides]AB[,]BC[and]CA[of the ABC triangle respectively (but they are not vertices). Let us denote with d_0, d_1, d_2, and d_3 the maximal side length of the DEF, DEA, DBF, CEF, triangles respectively.
Prove that $d_0 \geq (\surd t(3)/2) \min\{d_1, d_2, d_3\}$. When the equality takes place?

369.
The M set consists of k non-intersecting segments on the line. It is possible to put an arbitrary segment shorter than 1cm on the line in such a way, that his ends will belong to M. Prove that the total sum of the segment lengths is not less than 1/k cm.

370.
The infinite decimal notation of the real number x contains all the digits. Let v_n be the number of different n–digit segments encountered in x notation.
Prove that if for some n $v_n \leq (n+8)$, than x is a rational number.

The 18–th competition — Ashkhabad, 1984.

form	first day				second day			
8	371	372	373	374	383	384	385	386
9	375	376	377	378	387	388	389	390
10	379	380	381	382	391	392	393	394

371.
a) The product of n integers equals n, and their sum is zero. Prove that n is divisible by 4.
b) Let n is divisible by 4. Prove that there exist n integers such, that their product equals n, and their sum is zero.

372.
Prove that every positive a and b satisfy inequality
$$((a+b)^2)/2 + (a+b)/4 \geq a*\surd t(b) + b*\surd t(a).$$

373.
Given two equilateral triangles $A_1B_1C_1$ and $A_2B_2C_2$ in the plane. (The vertices are mentioned counterclockwise.) We draw vectors [OA[, [OB[, [OC[, from the arbitrary point O, equal to $[A_1A_2[$, $[B_1B_2[$, $[C_1C_2[$ respectively.
Prove that the triangle ABC is equilateral.

374.
Given four colours and enough square plates 1x1. We have to paint four edges of every plate with four different colours and combine plates, putting them with the edges of the same colour together.
Describe all the pairs m,n, such that we can combine those plates in a nxm rectangle, that has every edge of one colour, and its four edges have different colours.

375.
Prove that every positive x,y and real a satisfy inequality
$$x^{((\sin a)2)} * y^{((\cos a)2)} < x + y.$$

376.
Given a cube and two colours. Two players paint in turn a triple of arbitrary unpainted edges with his colour. (Everyone makes two moves.) The first wins if he has painted all the edges of some face with his colour. Can he always win?

377.

n natural numbers (n>3) are written on the circumference. The relation of the two neighbours sum to the number itself is a whole number. Prove that the sum of those relations is

a) not less than 2n;

b) less than 3n.

378.

The circle with the centre O is inscribed in the ABC triangle. The circumference touches its sides [BC], [CA], [AB] in A_1, B_1, C_1 points respectively. The [AO], [BO], [CO] segments cross the circumference in A_2, B_2, C_2 points respectively.

Prove that (A_1A_2),(B_1B_2) and (C_1C_2) lines intersect in one point.

379.

Find integers m and n such that $(5 + 3\ \sqrt{}t(2))^m = (3 + 5\ \sqrt{}t(2))^n$.

380.

n real numbers are written in increasing order in a line. The same numbers are written in the second line below in unknown order. The third line contains the sums of the pairs of numbers above from two previous lines. It comes out, that the third line is arranged in increasing order.

Prove that the second line coincides with the first one.

381.

Given ABC triangle. From the P point three lines (PA),(PB),(PC) are drawn. They cross the outscribed circumference in A_1, B_1,C_1 points respectively. It comes out that the $A_1B_1C_1$ triangle equals to the initial one.

Prove that there are not more than eight such a points P in a plane.

382.

Positive x,y,z satisfy a system:

$$x^2 + xy + (y^2)/3 = 25;$$
$$(y^2)/3 + z^2 = 9;$$
$$z^2 + zx + x^2 = 16.$$

Find the value of (xy + 2yz + 3zx) expression.

383.

The teacher wrote on a blackboard: " $x^2 + 10x + 20$ " Then all the pupils in the class came up in turn and either decreased or increased by 1 either the free coefficient or the coefficient at x, but not both. Finally they have obtained: " $x^2 + 20x + 10$ ".

Is it true that some time during the process there was written the square polynomial with the integer roots?

384.

The centre of the coin with radius r is moved along some polygon with the perimeter P, that is outscribed around the circle with radius R (R>r).

Find the coin trace area (a sort of polygon ring).

385.

There are scales and (n+1) weights with the total weight 2n. Each weight weight is an integer. We put all the weights in turn on the lighter side of the scales, starting from the heaviest one, and if the scales is in equilibrium — on the left side.

Prove that when all the weights will be put on the scales, they will be in equilibrium.

386.

Let us call "absolutely prime" the prime number, if having transposed its digits in an arbitrary order, we obtain prime number again.

Prove that its notation cannot contain more than three different digits.

387.
The x and y figures satisfy a condition: for every $n \geq 1$ the number xx...x6yy...y4 (n times x and n times y) is an exact square.
Find all possible x and y.

388.
The A,B,C and D points (from left to right) belong to the straight line.
Prove that every point E, that doesn't belong to the line satisfy:

$$|AE| + |ED| + |\,|AB| - |CD|\,| > |BE| + |CE|.$$

389.
Given a sequence $\{x_n\}$. $x_1 = x_2 = 1$. $x_{n+2} = (x_{n+1})^2 - \{x_n\}/2$.
Prove that the sequence has limit and find it.

390.
The white fields of 1983x1984 chess–board are filled with either +1 or –1. For every black field, the product of neighbouring numbers is +1. Prove that all the numbers are +1.

391.
The white fields of 3x3 chess–board are filled with either +1 or –1. For every field, let us calculate the product of neighbouring numbers. Then let us change all the numbers by the respective products.
Prove that we shall obtain only +1's, having repeated this operation finite number of times.

392.
What is more 2/201 or ln(101/100)? (No differential calculus allowed).

393.
Given three circles c_1,c_2,c_3 with r_1,r_2,r_3 radiuses, $r_1 > r_2$; $r_1 > r_3$. Each lies outside of two others. The A point — an intersection of the outer common tangents to c_1 and c_2 — is outside c_3. The B point — an intersection of the outer common tangents to c_1 and c_3 — is outside c_2. Two pairs of tangents — from A to c_3 and from B to c_2 — are drawn.
Prove that the quadrangle, they make, is outscribed around some circle and find its radius.

394.
Prove that every cube's cross–section, containing its centre, has the area not less then its face's area.

The 19–th competition — Mogilev, 1985.

form	first day				second day			
8	395	396	397	398	407	408	409	410
9	399	400	401	402	411	410	412	413
10	403	404	405	406	414	415	416	417

395.
Two perpendiculars are drawn from the middles of each side of the acute–angle triangle to two other sides. Those six segments make hexagon. Prove that the hexagon area is a half of the triangle area.

396.
Is there any number n, such that the sum of its digits in the decimal notation is 1000, and the sum of its square digits in the decimal notation is 1000000?

397.
What maximal number of the dames (? Am I right? The checker turns into a dame, when it reaches the last row?) can be put on the chess–board 8x8 so, that every dame can be taken by at least one other dame?

398.
You should paint all the sides and diagonals of the right n–angle so, that every pair of segments, having the common point, would be painted with different colours. How many colours will You require?

399.
Given a straight line and the point O out of the line. Prove that it is possible to move an arbitrary point A in the same plane to the O point, using only rotations around O and symmetry with respect to the l.

400.
The senior coefficient a in the square polynomial $P(x) = ax^2 + bx + c$ is more than 100. What is the maximal number of integer values of x, such that $|P(x)|<50$?

401.
Different natural numbers a,b,...,k are written in the diagram.

```
                k
                |
                V
           h--->i
           |    |
           V    V
      e--->f--->g
      |    |    |
      V    V    V
a---->b--->c--->d
```

Every number in the diagram equals to the sum of two numbers at the beginning of two arrows, leading to the number. What is the minimal possible value of d?

402.
Given unbounded strictly increasing sequence $a_1, a_2, \dots, a_n, \dots$ of positive numbers. Prove that
a) there exists a number k_0 such that for all $k>k_0$ the following inequality is valid:
$$a_1/a_2 + a_2/a_3 + \dots + a_k/a_{k-1} < k - 1$$
b) there exists a number k_0 such that for all $k>k_0$ the following inequality is valid:
$$a_1/a_2 + a_2/a_3 + \dots + a_k/a_{k-1} < k - 1985$$

403.
Find all the pairs (x,y) such that $|\sin(x)-\sin(y)| + \sin(x)\sin(y) \le 0$.

404.
The convex pentagon ABCDE was drawn in the plane.
A_1 was symmetric to A with respect to B.
B_1 was symmetric to B with respect to C.
C_1 was symmetric to C with respect to D.
D_1 was symmetric to D with respect to E.
E_1 was symmetric to E with respect to A.
How is it possible to restore the initial pentagon with the compasses and ruler, knowing A_1,B_1,C_1,D_1,E_1 points?

405.
The sequence $a_1, a_2, \dots, a_k, \dots$ is built according to the rules:
$$a_{2n} = a_n;\ a_{4n+1} = 1;\ a_{4n+3} = 0.$$
Prove that it is non-periodical sequence.

406.
n straight lines are drawn in a plane. They divide the plane onto several parts. Some of the parts are painted. Not a pair of painted parts has non-zero length common bound. Prove that the number of painted parts is not more than $(n^2 + n)/3$.

407.
Given a cube, a cubic box, that exactly suits for the cube, and six colours. First man paints each side of the cube with its (side's) unique colour. Another man does the same with the box.
Prove that the third man can put the cube in the box in such a way, that every cube side will touch the box side of different colour.

408.
The $[A_0A_5]$ diameter divides a circumference with the O centre onto two hemicircumferences. One of them is divided onto five equal arcs A_0A_1, A_1A_2, A_2A_3, A_3A_4, A_4A_5. The (A_1A_4) line crosses (OA_2) and (OA_3) lines in M and N points. Prove that $(|A_2A_3| + |MN|)$ equals to the circumference radius.

409.
If there are four numbers (a,b,c,d) in four registers of the calculating machine, they turn into (a–b,b–c,c–d,d–a) numbers whenever You press the button. Prove that if not all the initial numbers are equal, machine will obtain at least one number more than 1985 after some number of the operations.

410.
Numbers 1,2,3,...,2n are divided onto two equal groups. Let $a_1,a_2,...,a_n$ be the first group numbers in the increasing order, and $b_1,b_2,...,b_n$ — the second group numbers in the decreasing order.
Prove that

$$|a_1 - b_1| + |a_2 - b_2| + ... + |a_n - b_n| = n^2.$$

411.
The parallelepiped is constructed of the equal cubes. Three parallelepiped faces, having the common vertex are painted. Exactly half of all the cubes have at least one face painted.
What is the total number of the cubes?

412.
One of two circumferences of radius R comes through A and B vertices of the ABCD parallelogram. Another comes through B and D. Let M be another point of circumferences intersection.
Prove that the circle outscribed around AMD triangle has radius R.

413.
Given right hexagon. The lines parallel to all the sides are drawn from all the vertices and middles of the sides (consider only the interior, with respect to the hexagon, parts of those lines). Thus the hexagon is divided onto 24 triangles, and the figure has 19 nodes. 19 different numbers are written in those nodes. Prove that at least 7 of 24 triangles have the property: the numbers in its vertices increase (from the least to the greatest) counterclockwise.

414.
Solve the equation ("2" encounters 1985 times):

$$\cfrac{x}{2 + \cfrac{x}{2 + \cdots \cfrac{x}{2 + \cfrac{x}{1 + \sqrt{(1 + x)}}}}} = 1$$

415.
All the points situated more close than 1cm to ALL the vertices of the right pentagon with 1cm side, are deleted from that pentagon. Find the area of the remained figure.

416.
Given big enough sheet of cross–lined paper with the side of the squares equal to 1. We are allowed to cut it along the lines only. Prove that for every m>12 we can cut out a rectangle of the greater than m area such, that it is impossible to cut out a rectangle of m area from it.

417.
The $ABCDA_1B_1C_1D_1$ cube has unit length edges. Find the distance between two circumferences, one of those is inscribed into the ABCD base, and another comes through A,C and B_1 points.

The 20–th competition — Ulyanovsk, 1986.

form	first day				second day			
8	418	419	420	421	430	431	432	433
9	422	423	424	425	434	433b	435	436
10	426	427	428	429	437	438	439	440

418.
The square polynomial $x^2+ax+b+1$ has natural roots.
Prove that (a^2+b^2) is a composite number.

419.
Two equal squares, one with red sides, another with blue ones, give an octagon in intersection. Prove that the sum of red octagon sides lengths is equal to the sum of blue octagon sides lengths.

420.
The point M belongs to the [AC] side of the acute–angle triangle ABC. Two circles are outscribed around ABM and BCM triangles. What M position corresponds to the minimal area of those circles intersection?

421.
Certain king of a certain state wants to build n cities and n–1 roads, connecting them to provide a possibility to move from every city to every city. (Each road connects two cities, the roads do not intersect, and don't come through another city.) He wants also, to make the shortests distances between the cities, along the roads, to be 1,2,3,...,n(n–1)/2 kilometres. \Is it possible for

a) n=6;
b) n=1986?

422.
Prove that it is impossible to draw a convex quadrangle, with one diagonal equal to doubled another, the angle between them 45 degrees, on the coordinate plane, so, that all the vertices' coordinates would be integers.

423.
Prove that the rectangle mxn table can be filled with exact squares so, that the sums in the rows and the sums in the columns will be exact squares also.

424.
Two circumferences, with the distance d between centres, intersect in P and Q points. Two lines are drawn through the A point on the first circumference (Q<>A<>P) and P and Q points. They intersect the second circumference in the B and C points.

a) Prove that the radius of the circle, outscribed around the ABC triangle, equals d.
b) Describe the set of the new circle's centres, if the A point moves along all the first circumference.

425.
Given right hexagon. Each side is divided onto 1000 equal segments. All the points of division are connected with the segments, parallel to sides. Let us paint in turn the triples of unpainted nodes of obtained net, if they are vertices of the unilateral triangle, doesn't matter of what size an orientation. Suppose, we have managed to paint all the vertices except one.
Prove that the unpainted node is not a hexagon vertex.

426.
Find all the natural numbers equal to the square of its divisors number.

427.
Prove that the following inequality holds for all positive $\{a_i\}$.

$$1/a_1 + 2/(a_1+a_2) + ... + n/(a_1+...+a_n) < 4(1/a_1 + ... + 1/a_n)$$

428.
A line is drawn through the A vertex of ABC triangle with $|AB|<>|AC|$. Prove that the line can not contain more than one point M such, that M is not a triangle vertex, and the angles ABM abd ACM are equal. What lines do not contain such a point M at all?

429.
A cube with edge of length n ($n\geq 3$) consists of n^3 unit cubes. Prove that it is possible to write different n^3 integers on all the unit cubes to provide the zero sum of all integers in the every row parallel to some edge.

430.
The decimal notation of three natural numbers consists of equal digits: n digits x for a, n digits y for b and 2n digits z for c.
For every $n > 1$ find all the possible triples of digits x,y,z such, that $a^2 + b = c$.

431.
Given two points inside a convex dodecagon (twelve sides) situated 10cm far from each other. Prove that the difference between the sum of distances, from the point to all the vertices, is less than 1m for those points.

432.
Given 30 equal cups with milk. An elf tries to make the amount of milk equal in all the cups. He takes a pair of cups and aligns the milk level in two cups. Can there be such an initial distribution of milk in the cups, that the elf will not be able to achieve his goal in a finite number of operations?

433.
Find the relation of the black part length and the white part length for the main diagonal of the
a) 100x99 chess–board;
b) 101x99 chess–board.

434.
Given right n–angle $A_1A_2...A_n$. Prove that if
a) n is even number, than for the arbitrary point M in the plane, it is possible to choose signs in an expression

$$+-[MA_1[\ +-[MA_2[\ +- ...\ +-[MA_n[$$

to make it equal to the zero vector (here [MA[denotes vector).
b) n is odd, than the abovementioned expression equals to the zero vector for the finite set of M points only.

435.
All the fields of a square nxn (n>2) table are filled with +1 or –1 according to the rules:
At the beginning –1 are put in all the boundary fields. The number put in the field in turn (the field is chosen arbitraryly) equals to the product of the closest, from the different sides, numbers in its row or in its column.
a) What is the minimal
b) What is the maximal possible number of +1 in the obtained table?

436.
Prove that for every natural n the following inequality is valid:

$$|\sin 1| + |\sin 2| + |\sin(3n-1)| + |\sin(3n)| > 8n/5.$$

437.
Prove that the sum of all numbers representable as $1/(mn)$, where m,n — natural numbers, $1 \le m < n \le 1986$; is not an integer.

438.
A triangle and a square are outscribed around the unit circle. Prove that the intersection area is more than 3.4. Is it possible to assert that it is more than 3.5?

439.
Let us call a polynomial "admissible" if all it's coefficients are 0, 1, 2 or 3. For given n find the number of all the admissible polynomials P such, that $P(2) = n$.

440.
Consider all the tetrahedrons AXBY, outscribed around the sphere. Let A and B points be fixed. Prove that the sum of angles in the non-plane quadrangle AXBY doesn't depend on X and Y points.

The 21–st competition — Frunze, 1987.

form	first day				second day			
8	441	442	443	444	452	453	454	455
9	445	446a	447	448	456	457	458	459
10	449	450	451	446b	455	460	461	462

441.
Ten sportsmen have taken part in a table–tennis tournament (each pair has met once only, no draws). Let x_i be the number of i–th player victories, y_i — losses.
Prove that $x_1^2 + ... + x_{10}^2 = y_1^2 + ... + y_{10}^2$

442.
It is known that, having 6 weighs, it is possible to balance the scales with loads, which weights are successing natural numbers from 1 to 63.
Find all such sets of weighs.

443.
Given right heptagon (7–angle) $A_1...A_7$.
Prove that $1/|A_1A_5| + 1/|A_1A_3| = 1/|A_1A_7|$.

444.
The "Sea battle" game.

a) You are trying to find the 4–field ship — a rectangle 1x4, situated on the 7x7 playing board. You are allowed to ask a question, whether it occupies the particular field or not. How many questions is it necessary to ask to find that ship surely?
b) The same question, but the ship is a connected (i.e. its fields have common sides) set of 4 fields.

445.
Prove that $(1^{1987} + 2^{1987} + ... + n^{1987})$ is divisible by (n+2).

446.
```
+--+
|  |
+--+--+
|  |  |
+--+--+
```
a) How many such figures can You put on a 8x8 board without intersections?
b) An arbitrary field is cut out of a 1987x1987 board. Prove that the reminding part of the board can be always cut on such parts.

447.
Three lines are drawn parallel to the sides of the triangles in the opposite to the vertex, not belonging to the side, part of the plane. The distance from each side to the corresponding line equals the length of the side. Prove that six intersection points of those lines with the continuations of the sides are situated on one circumference.

448.
Given two closed broken lines in the plane with odd numbers of edges. All the lines, containing those edges are different, and not a triple of them intersects in one point.
Prove that it is possible to chose one edge from each line such, that the chosen edges will be the opposite sides of a convex quadrangle.

449.
Find a set of five different relatively prime natural numbers such, that the sum of an arbitrary subset is a composite number.

450.
Given a convex pentagon. The angles ABC and ADE are equal. The angles AEC and ADB are equal too.
Prove that the angles BAC and DAE are equal also.

451.
Prove such a, that all the numbers cos a, cos 2a, cos 4a, ... , $\cos((2^n)a)$ are negative.

452.
The positive numbers a,b,c,A,B,C satisfy a condition a + A = b + B = c + C = k.
Prove that $aB + bC + cA \le k^2$.

453.
Each field of the 1987x1987 board is filled with numbers, which absolute value is not greater than one. The sum of all the numbers in every 2x2 square equals 0.
Prove that the sum of all the numbers is not greater than 1987.

454.
The B vertex of the ABC angle lies out the circle, and the [BA) and [BC) beams intersect it. The K point belongs to the intersection of the [BA) beam and the circumference. The KP chord is orthogonal to the

angle ABC bisector. The (KP) line intersects the BC beam in the M point. Prove that the [PM] segment is twice as long as the distance from the circle centre to the angle ABC bisector.

455.
Two players are writting in turn natural numbers not exceeding p. The rules forbid to write the divisors of the numbers already having been written. Those who cannot make his move looses.
a) Who, and how, can win if p=10?
b) Who wins if p=1000?

456.
Every evening uncle Chernomor (see Pushkin's tales) appoints either 9 or 10 of his 33 "knights" /* let's call them so */ in the "night guard" /* let's call it so */. When it can happen, for the first time, that every knight has been on duty the same number of times?

457.
Some points with the integer coordinates are marked on the coordinate plane. Given a set of nonzero vectors. It is known, that if You apply the beginnings of those vectors to the arbitrary marked point, than there will be more marked ends of the vectors, than not marked.
Prove that there is infinite number of marked points.

458.
The convex n–angle ($n \geq 5$) is cut along all its diagonals.
Prove that there are at least a pair of parts with the different areas.

459.
The T_0 set consists of all the numbers, representable as $(2^k)!$, k = 0, 1, 2, ... , n, ... The T_m set is obtained from T_{m-1} by adding all the finite sums of different numbers, that belong to T_{m-1}. Prove that there is a natural number, that doesn't belong to T_{1987}.

460.
The plot of the y=f(x) function, being rotated by the (right) angle around the (0,0) point is not changed.
a) Prove that the equation f(x)=x has the unique solution.
b) Give an example of such a function.

461.
All the faces of a convex polyhedron are the triangles. Prove that it is possible to paint all its edges in red and blue colour in such a way, that it is possible to move from the arbitrary vertex to every vertex along the blue edges only and along the red edges only.

462.
Prove that for every natural n the following inequality is held:

$$(2n + 1)^n \geq (2n)^n + (2n - 1)^n.$$

From the Holy Land, with respect:


```
 /\ /\   Vladimir A. Pertsel            S/W  engineer
((ovo))        E-mail:          \|/   Sagantec Israel
():::() voldemar@sagantec.co.il (o o) tel.972-4-8572781
--PVA-----------------------ooO-(_)-Ooo------------------
```


an ancestor of mine by the name of Noah was once the commanding admiral of the combined fleets of my planet.

Solutions:

1.

```
A-----B-----C
|     |     |
D--E--F--G--H
|  |     |  |
I--J-----K—-L
```

If a curve intersects the boundary of a region R (such as ABFED), then it moves from inside R to outside or vice versa. Hence if R has an odd number of edges (like ABFED) then a curve intersecting all of them just once must have one endpoint inside R. But there are four such regions (ABFED, BCHGF, EFGKJ and the outside of ABCHLKJID) and only two endpoints.
Note that we can easily intersect all edges but one. For example, start above AB, then cross successively AB, AD, DI, DE, EF, EJ, IJ, JK, GK, KL, HL, GH, CH, BC, FG.

2.
Let O be the center of the rectangle. Let $r = (a+c)/2 = (b+d)/2$. The required circle has center O, radius r. Let an outer common tangent touch the circle center A at W, and the circle center C at X. Let P be the midpoint of WX, then OP is parallel to AW and CX and has length r, hence the circle center O touches AW at P. Similarly for the other common tangents.

3.
Let n be the smallest number in the sequence and m the smallest with last digit 0. m and m+10 have different digit sums unless (possibly) the penultimate digit of m is 9, but in that case m+10 and m+20 have different digit sums. So two of m, m+10, m+20 are sure to have different digit sums. Hence at least one has a digit sum not congruent to 1 mod 11. Adding the appropriate final digit gives a number whose digit sum is divisible by 11. This number lies in the range m to m+29 and $m \le n+9$. Hence the result. n=999981 shows it is best possible.

4.
(a)

```
**..
*.*.
.**.
...*
```

Pick any two rows. The unpicked stars lie in different columns.

(b) If there is a row with at least 3 stars, pick it. That leaves at most 3 stars, pick the row for one and the columns for the others. Now assume no row has more than 2 stars. 6 stars in <6 rows, so we can pick a row with 2 stars. That leaves 4 stars in 3 rows, so we can pick another row with 2 stars. That leaves 2 stars. Pick their columns. [This glosses over the case of <6 stars. In this case we can add extra stars to make the number up to 6. Now the procedure above deals with the original stars and the extra stars, and in particular with the original stars.]

5.
a) Let Q_0 be the original quadruple (a, b, c, d) and Q_n the quadruple after n transformations. If abcd>1, then the products form a strictly increasing sequence, so return is impossible. Similarly if abcd<1. So we must have abcd=1. Let the largest of the four values of a quadruple Q be M(Q). If a member of Q_1 is not 1, then $M(Q_1)>1$. Q_3 consists of the elements of Q_1 squared and permuted, so $M(Q_3) = M(Q_1)^2$. Hence the sequence $M(Q_1)$, $M(Q_3)$, $M(Q_5)$, ... increases without limit. This means no return is possible, because a return would lead to the values cycling.

(b) After r<n transformations, the first number of the n–tuple is the product $a_1^{(r|0)}a_2^{(r|1)} \ldots a_{r+1}^{(r|r)}$, where (r|i) denotes the binomial coefficient. [This is an easy induction.] Hence after $n=2^k$ transformations it is a_1^2 times the product $a_2^{(n|1)} \ldots a_n^{(n|n-1)}$. So it is sufficient to prove that (n|i) is even for n a power of 2 and 0<i<n. But observe that (n|i) = (n–1|i) n/(n–i) and n is divisible by a higher power of 2 than n–i.

6.

(a) Represent A, B as complex numbers $z_1 + w_1e^{it}$, $z_2 + w_2e^{it}$. Then C is $(z_1 + w_1e^{it}) + (z_2 + w_2e^{it} - z_1 - w_1e^{it}) e^{i\pi/3}$, which is also of the form $z + w\, e^{it}$.

However, there is one subtlety. There are actually two circles possible for C depending on which side of AB we place it. The continuity requirement means that C is normally confined to one of the circles. However, if A and B ever coincide then C may be able to switch to the other circle.
If we regard "moves continuously" as allowing a discontinuous velocity, then a switch is always possible (provided A and B coincide).

(b) Answer: 5.

7.

The array has mn entries. Call an array that can be obtained by repeated changes a reachable array. A reachable array differs from the original only in that some or all of the signs of its mn entries may be different. There are at most 2 possibilities for each sign and hence at most 2^{mn} different reachable arrays. For each reachable array calculate the sum of all its entries. Take the reachable array with the largest such sum. It must have non-negative row and column sums, because if any such sum was negative, changing the sign of that row or column would give another reachable array with strictly greater total sum.

8.

Every point must have at least one edge. We show that there is a point with just one edge. Suppose the contrary, that every point has at least two edges. We now construct a path in which the same edge or point never appears twice. Starting from any point b, move along an edge to c. c is not already on the path, because otherwise the edge would join b to itself. Now suppose we have reached a point x not previously on the path. x has at least two edges, so it must have another one besides the one we used to reach it. Suppose this joins x to y. If y is already on the path, then we have two distinct ways of moving along edges from x to y: directly, or by backtracking along the path from x to y. But this is impossible, so y is not already on the path and we may extend the path to it. But this procedure allows us to construct a path containing more than the n distinct points available. Contradiction.

9.

Care is needed. Although easy, this is more awkward than it looks.
Let d=(m,n), the greatest common divisor of m and n. Let r=n/d, s=nhk – m/d, where h is any integer sufficiently large to ensure that s>0. Now rm+sn = mn/d + nnhk – mn/d = nnhk, which is a multiple of k. If e divides r, then it also divides rdhk = nhk. So if e divides r and s, then it also divides s – nhk = –m/d. But n/d and m/d are relatively prime, so e must be 1. Hence r and s are relatively prime.

10.

Answers: [N/2], [(N+1)/2], [N/2].
Suppose A leaves piles n, m with n≤m. Under *R1*, B can certainly secure m by dividing the larger pile into 1 and m–1. He cannot do better, because if b is the biggest of the 4 piles, then the smallest is at most m–b. Hence A's best strategy is to leave [N/2], [(N+1)/2].
Under *R2*, if A leaves a=2, b=N–2, then B cannot do better than [N/2], because if he divides the larger pile into a,b with a≤b, then he takes a+1. A cannot do better, because if he leaves a,b with 3≤a≤b, then B can divide to leave 1, a–1, [b/2], [(b+1)/2]. Now if a–1≥[(b+1)/2], then B takes b≥[(N+1)/2]. If a–1<[(b+1)/2], then B takes a–1+[b/2]. But a–1≥2 and [b/2]≥[(b+1)/2]–1, so a–1+[b/2]≥1+[(b+1)/2], or B takes at least as many as A, so B takes at least [(N+1)/2].

Under *R3*, A's best strategy is to divide into [N/2],[(N+1)/2]. We have already shown that B can secure [(N+1)/2] and no more by following *R1*. He cannot do better under R2, for if he divides so that the biggest pile comes from [N/2], then the smallest does too and so he gets [(N+1)/2]. If he divides so that the biggest and smallest piles come from [(N+1)/2], then he gets only [N/2]. But one of these must apply, because if he divided so that the smaller from [N/2] was smaller than the smaller from [(N+1)/2], and the bigger from [N/2] was smaller than the bigger from [(N+1)/2], then [N/2] would be at least 2 less than [(N+1)/2] (which it is not).

11.
Given any infinite sequence of natural numbers we can find a non-decreasing subsequence (proof below). So suppose the three sequences are a_i, b_i, and c_i. Take a non-decreasing subsequence of a_i. Suppose it is a_{i1}, a_{i2}, a_{i3}, ... Now consider the infinite sequence b_{i1}, b_{i2}, ... It must have a non-decreasing subsequence. Suppose it is b_{j1}, b_{j2}, ... Now consider the infinite sequence c_{j1}, c_{j2}, ... It must have a non-decreasing subsequence c_{k1}, c_{k2}, ... Each of the three sub–sequences a_{k1}, a_{k2}, ... , b_{k1}, b_{k2}, ... , c_{k1}, c_{k2}, ... is non-decreasing. So we may take, for example, m=k2 and n=k1.
[Proof that any infinite sequence of natural numbers has a non-decreasing subsequence: if the original sequence is unbounded, then we can take a strictly increasing subsequence. If not, then since there are only finitely many possible numbers not exceeding the bound, at least one of them must occur infinitely often.]

12.
If a circle with unit diameter intersects a unit square, then its center must lie inside an area $3+\pi/4$, namely an oval centered on the square and comprising: the original square, area 1; four 1 x 1/2 rectangles on the sides, total area 2; and four quarter circles at the corners, total area $\pi/4$. So if it does not intersect any of the 120 unit squares, then it must avoid ovals with a total area of 120 x $(3+\pi/4)$ = 454.2. Of course, for many arrangements of the squares, these ovals might overlap substantially, but the worst case would be no overlap. The circle is also required to lie inside the rectangle, so its center must lie outside a strip 1/2 wide around the edge, and hence inside an inner 19 x 24 rectangle, area 456. The total area of ovals is less, so they cannot cover it completely and it must be possible to place a circle as required.

13.
Compare the triangles A'B'A and ADB. The base of A'B'A can be taken as A'A, which is the same length as AD. The height of A'B'A is AB' times *sin* B'AA', which is twice AB times *sin* BAD. So area A'B'A = 2 area ADB. Similarly, area B'C'B = 2 area BAC, area C'D'C = 2 area CBD, and area D'A'D = 2 area DCA. So adding, the area A'B'A + area C'D'C = 2 area ABCD, and area B'C'B + area D'A'D = 2 area ABCD. But ABCD = A'B'A + B'C'B + C'D'C + D'A'D + ABCD. Hence result.

14.
Let the common tangent meet C at S. Let X be the intersection of C and OP lying between O and P. PT = PO, hence angle POT = angle PTO, so angle OPT = 180 – 2 angle POT. But PT and OS are parallel, because both are perpendicular to the common tangent. Hence angle POS = 2 angle POT, so angle SOT = angle XOT. Hence TX is tangent to C, in other words T lies on the (fixed) tangent to C at X. Conversely, it is easy to see that any such point can be obtained (just take P such that PO = PT). Thus the required locus is the pair of tangents to C which are perpendicular to L.

15.
An easy induction gives $a_r = (2^r - 1)a_1 - (2^r - 2)a_0$ for r = 2, 3, ... , 100. Hence, in particular, $a_{100} = (2^{100} - 2)(a_1 - a_0) + a_1$. But a_1 and $(a_1 - a_0)$ are both at least 1. Hence result.

16.
If there were such values, then subtract the equation with x = 19 from the equation with x = 62 to get: $a(62^3 - 19^3) + b(62^2 - 19^2) + c(62 - 19) = 1$. But the left hand side is divisible by 62 –19 = 43, contradiction.

17.
If we change a –1 to 1, we affect the total number of rows and columns (containing an odd number of –1s) by 0, 2 or –2. After changing all the –1s we have total of 0. Hence the starting total must be even. So it cannot be n.

18.
Let M be the midpoint of AB and X the midpoint of MB. Construct the circle center B, radius BC/2 and the circle diameter AX. If they do not intersect (so BC<AB/2 or BC>AB) then the construction is not possible. If they intersect at N, then take C so that N is the midpoint of BC. Let CM meet AN at O. Then AO/AN = AM/AX =2/3, so the triangles AOM and ANX are similar. Hence angle AOM = angle ANX = 90.

19.
Applying the arithmetic/geometric mean result to the 10 numbers gives the result immediately.

20.
Let X be the midpoint of AB and O the center of ABCDE. Suppose M lies inside AXO. Then $ME=r_3$. So we maximise r_3 by taking M at X, with distance 1.5590, and we minimise r_3 by taking M as the intersection of AO and EB with distance 0.8090. AXO is one of 10 congruent areas, so the required loci are (a) the 5 midpoints of the diagonals, and (b) the 5 midpoints of the sides.

21.
$x \le 9.1998 = 17982$. Hence $y \le$ the greater of 1+7+9+9+9=35 and 9+9+9+9=36. But 9 divides the original number and hence also x, y and z. Hence z=9.

22.
Take X on AH so that BX is perpendicular to AH. Extend to meet HM at P'. Let N be the midpoint of AB. A, B, M and X are on the circle center N radius NA (because angles AMB and AXB are 90). Also MN is parallel to BC (because AMN, ACB are similar), so NM is perpendicular to MH, in other words HM is a tangent to the circle. hence P'M.P'M = P'X.P'B. Triangles P'XH and P'HB are similar (angles at P' same and both have a right angle), so P'H/P'X = P'B/P'H, so P'H.P'H = P'X.P'B. Hence P'H = P'M and P' coincides with P.

23.
If we ignore the restrictions of CA, then the maximum area is 1, achieved when AB is perpendicular to BC. But in this case CA satisfies the restrictions.

24.
Put x–y=r, y–z=s. Then z–x = –(r+s), and $(x-y)^5 + (y-z)^5 + (z-x)^5 = r^5 + s^5 - (r+s)^5 = -5r^4s - 10r^3s^2 - 10r^2s^3 - 5rs^4 = -5rs(r+s)(r^2 + rs + s^2)$.

25.
The essential point is that if we plot the values a_r against r, then the curve formed by joining the points is cup shaped. Its two endpoints are on the axis, so the other points cannot be above it.There are many ways of turning this insight into a formal proof. Barry Paul's was neater than mine: $a_{r+1}-a_r \ge a_r-a_{r-1}$. Hence (easy induction) if $a_s - a_{s-1} > 0$, then $a_n > a_s$. Take a_s to be the first positive, then certainly $a_s > a_{s-1}$, so $a_n > 0$. Contradiction.

26.
Induction on m+n. Trivial for m+n=2. Let x be the largest number in the two given sets. Suppose it is a row total; let y be the largest column total. If y<x, then replace x by x–y in the set of row totals and remove y from the col totals. By induction find ≤m+n–2 positive numbers in an m x (n–1) array with the new totals. Adding a col empty except for y in the row totalling x–y gives the required original set. If y=x, then drop x from the row totals and y from the col totals and argue as before. If x was a col total we interchange rows and cols in the argument.

27.
Let the circles be a, b, c, d, e. Let A be a point common to b, c, d, e, let B be a point common to a, c, d, e and so on. If any two of A, B, C, D, E coincide then the coincident point is on all 5 circles. Suppose they are all distinct. Then A, B, C are on d and e. Hence d and e coincide (3 points determine a circle). Hence D is on all 5 circles.

28.
The bottom 4 played 6 games amongst themselves, so their scores must total at least 6. Hence the number 2 player scored at least 6. The maximum score possible is 7, so if the number 2 player scored more than 6, then he must have scored 6 1/2 and the top player 7. But then the top player must have won all his games, and hence the number 2 player lost at least one game and could not have scored 6 1/2. Hence the number 2 player scored exactly 6, and the bottom 4 players lost all their games with the top 4 players. In particular, the number 3 player won against the number 7 player.

29.
(a) Let the quadrilateral be ABCD and let the diagonals AC, BD meet at E. Then area ABC = AC.EB.*sin* CEB/2, and area ADC = AC.ED.*sin* CEB/2, so E is the midpoint of BD. Similarly, it is the midpoint of AC. Hence the triangles AEB and CED are congruent, so angle CDE = angle ABE, and hence AB is parallel to CD. Similarly, AD is parallel to BC.

(b) Let the hexagon be ABCDEF. Let BE, CF meet at J, let AD, CF meet at K, and let AD, BE meet at L. Let AK=a, BJ=b, CJ=c, DL=d, EL=e, FK=f. Also let KL=x, JL=y and JK=z. Consider the pair of diagonals AD, BE. They divide the hexagon into 4 parts: the triangles ALB and DLE, and the quadrilaterals AFEL and BCDL. Since area ALB + area AFEL = area DLE + area BCDL, and area ALB + area BCDL = area DLE + area AFEL, the two triangles must have the same area (add the two inequalities). But area ALB = 1/2 AL.BL.*sin* ALB, and area DLE = 1/2 DL.EL.*sin* DLE = 1/2 DL.EL.*sin* ALB, so AL.BL = DL.EL or de = (a+x)(b+y). Similarly, considering the other two paris of diagonals, we get bc = (e+y)(f+z) and af = (c+z)(d+x). Multiplying the three inequalities gives: abcdef = (a+f)(b+y)(c+z)(d+x)(e+y)(f+z). But x, y, z are non-negative, so they must be zero and hence the three diagonals pass through a common point.

30.
If d divides m+n and m^2+n^2, then it also divides $(m+n)^2 - (m^2+n^2) = 2mn$ and hence also $2m(m+n) - 2mn = 2m^2$ and $2n(m+n) - 2mn = 2n^2$. But m and n are relatively prime, so m^2 and n^2 are also. Hence d must divide 2.

31.
(a) Take Y on the circle so that angle ABY=90. Then AY is a diameter and so angle AMY=90. Take X as the midpoint of BY. Then triangles BXK and BYM are similar, so XK is parallel to YM. Hence XK is perpendicular to AM, and so P is the intersection of XK and AM. In other words, KP always passes through X.
(b) P must lie on the circle diameter AX, and indeed all such points can be obtained (given a point P on the circle, take M as the intersection of AP and the original circle). So the locus of P is the circle diameter AX.

32.
Answer: 2/3.
Let O be the center of ABC. Let AO meet BC at D, let BO meet CA at E, and let CO meet AB at F. Given any point X inside ABC, it lies in one of the quadrilaterals AEOF, CDOE, BFOD. Without loss of generality, it lies in AEOF. Take the line through X parallel to BC. It meets AB in P and AC in Q. Then PQ is shorter than the parallel line MON with M on AB and N on AC, which has length 2/3. If we twist the segment PXQ so that it continues to pass through X, and P remains on AB and Q on AC, then its length will change continuously. Eventually, one end will reach a vertex, whilst the other will be on the opposite side and hence the length of the segment will be at least that of an altitude, which is greater than 2/3. So at some intermediate position its length will be 2/3.

To show that no value smaller than 2/3 is possible, it is sufficient to show that any segment POQ with P and Q on the sides of the triangle has length at least 2/3. Take P on MB and Q on AN with P, O, Q collinear. Then PQ cos POM = MN – QN cosπ/3 + PM cos π/3. But PM>QN (using the sine rule, PM = OM sinPOM/sinOPM and QN = ON sinQON/sinOQN, but OM=ON, angle POM = angle QON, and angle OQN = angle OPM + π/3 > angle OPM), and hence PQ > MN sec POM > MN.

33.
a) We say a domino bridges two columns if half the domino is in each column. We show that for 0<n<6 the number of dominoes bridging columns n and n+1 must be at least 2 and even.
Consider first n=1. There cannot be 3 dominoes entirely in column 1, or it would be separately tiled. So there must be at least one domino bridging columns 1 and 2. The number must be even, because it must equal the number of squares in column 1 (even) less twice the number of dominoes (entirely) in column 1.
Now suppose it is true for n<5 and consider column n+1. There must be at least one domino bridging columns n+1 and n+2, or columns 1 thru n+1 would be separately tiled. The number must be even, because it must equal the number of squares in column n+1 (even) less the number bridging n and n+1 (even) less twice the number entirely in the column.
So in total there are at least 5 x 2 = 10 dominoes bridging columns. By the same argument there are at least another 10 bridging rows, but there are only 18 dominoes in total.

(b) No. For example:

```
1 2 3 3 1 1 2 2
1 2 1 2 2 3 3 1
3 3 1 3 1 2 4 1
1 2 2 3 1 2 4 3
1 3 3 2 2 1 2 3
3 2 1 1 4 1 2 1
3 2 3 2 4 3 3 1
1 1 3 2 1 1 2 2
```

34.
Assume $a_1 < a_2 < ... < a_n$. We have the following collection of increasing sums:

$a_1 < a_2 < ... < a_n$	n sums
$a_1+a_n < a_2+a_n < ... < a_{n-1}+a_n$	n–1 sums
$a_1+a_{n-1}+a_n < a_2+a_{n-1}+a_n < ... < a_{n-2}+a_{n-1}+a_n$	n–2 sums
...	
$a_1 + a_2 + ... + a_n$	1 sum

A total of 1+2+ ... +n = n(n+1)/2.

36.
Let the square be a^2 and the difference d, so that all numbers of the form a^2+nd belong to the arithmetic progression (for n a natural number).
Take n to be $2ar + dr^2$, then $a^2+nd = (a+dr)^2$.

37.
10 x 5 > 45, so some digit i_0 must appear less than 5 times. But each occurrence can give at most 2 edges i_0, j, so there are at most 8 edges i_0, j, which is one too few.

38.
Comparing coefficients of x^{20}, we must have $a = (2^{20} - 1)^{1/20}$ (note that we allow either the positive or the negative root).
Set x=1/2. Then we must have $(ax + b)^{20} = 0 = (x^2+px+q)^{10}$, and hence ax + b =0 and $x^2+px+q = 0$. So b = –a/2, and 1/4 + p/2 + q = 0.
Set x=0. Then we get $q^{10} = 1 - b^{20} = 1/2^{20}$, so q = 1/4 or –1/4, and p= –1 or 0 respectively. Comparing the coefficients of x^{19}, we must have p = –1 and q = 1/4. So, if there is a solution, then it must be: $a = (2^{20} -$

$1)^{1/20}$, $b = -a/2$, $p = -1$, $q = 1/4$. This is indeed a solution because with these values, the lhs $= 2^{20}(x - 1/2)^{20} - (x - 1/2)^{20}a^{20} = (x - 1/2)^{20} = (x^2 - x + 1/4)^{10}$ = rhs.

39.
Answer: 2.3^{n-1}.
True for n=1. The new points added at step n+1 have twice the sum of the points after step n, because each old point contributes to two new points. hence the total after step n+1 is three times the total after step n.

40.
Let the triangle be ABC, with AB=AC. Take the circle through B and C which has AB and AC as tangents. The required locus is the arc BC.
Suppose P lies on the arc. Let the perpendiculars from P meet BC in L, AB in N and AC in M. Join PB and PC. The triangles PNB and PLC are similar (PNB and PLC are both 90, and NBP = LCP because NB is tangent to the circle). Hence PN/PL = PB/PC. Similarly, triangles PMC and PLB are similar and hence PM/PL = PC/PB. Multiplying gives the required result $PL^2 = PM.PN$.
If P is inside the circle and not on it, take P' as the intersection of the line AP and the arc. We have PL<P'L, but PM>P'M and PN>P'N, hence $PL^2 < PM.PN$. Similarly, if P is outside the circle and not on it, then $PL^2 > PM.PN$.

41.
Let k be twice the area of the triangle. Then $k \geq BC^2$, $k \geq AC^2$ and $k \leq AC.BC$, with equality in the last case only if AC is perpendicular to BC. Hence AC and BC have equal lengths and are perpendicular. So the angles are 90, 45, 45.

42.
m and m+1 have no common divisors, so each must separately be an nth power. But the difference betwee the two nth powers is greater than 1 (for n>1).

43.
Taking digit sums repeatedly gives the remainder after dividing the number by 9, or 9 if the number is exactly divisible by 9. $10^9 - 1 = 9n$, and for any $r \geq 0$ the nine consecutive numbers 9r+1, 9r+2, ... , 9r+9 include just one number giving remainder 1 and one number giving remainder 2. Hence the numbers up to $10^9 - 1$ give equal numbers of 1s and 2s. 10^9 itself gives 1, so there is just one more of the 1s than the 2s.

44.
Let the smallest value be s and suppose it occurs m times (with m<n). Then the values in the next stage are all at least s, and at most m–1 equal s. So after at most m iterations the smallest value is increased.
We can never reach a stage where all the values are equal, because if $(a_1+a_2)/2 = (a_2+a_3)/2 = ... = (a_{n-1}+a_n)/2 = (a_n+a_1)/2$, then $a_1+a_2 = a_2+a_3$ and hence $a_1 = a_3$. Similarly, $a_3 = a_5$, and so $a_1 = a_3 = a_5 = ... = a_n$ (n odd). Similarly, $a_2 = a_4 = ... = a_{n-1}$. But we also have $a_n + a_1 = a_1 + a_2$ and so $a_2 = a_n$, so that all a_i are equal. In other words, if all the values are equal at a particular stage, then they must have been equal at the previous stage, and hence at every stage.
Thus if the values do not start out all equal, then the smallest value increases indefinitely. But that is impossible, because the sum of the values is the same at each stage, and hence the smallest value can never exceed $(a_1 + ... + a_n)/n$.
Note that for n even the argument breaks down because a set of unequal numbers can iterate into a set of equal numbers. For example: 1, 3, 1, 3, ... , 1, 3.

45.
(a) Extend AB, CD, EF. We get an equilateral triangle with sides AF + AB + BC, BC + CD + DE, ED + EF + FA. Hence AB – DE = CD – FA = EF – BC, as required.
(b) Take an equilateral triangle with sides s, t, u lengths $a_2 + a_3 + a_4$, $a_4 + a_5 + a_6$, and $a_6 + a_1 + a_2$ respectively. Construct BC length a_2 parallel to t with B on u and C on s. Construct DE length a_4 parallel to u with D on s and E on t. Construct FA length a_6 parallel to s with F on t and A on u. Then ABCDEF is the required hexagon, with AB = a_1, BC = a_2 etc.

46.

Let $s_1 = \sqrt{t}(x)$, $s_2 = \sqrt{t}(x + s_1)$, $s_3 = \sqrt{t}(x + s_2)$ and so on. So the equation given is $y = s_{1998}$. We show first that all s_n must be integral for $1 \le n \le 1998$. y is integral, so s_{1998} is integral. Now suppose s_n is integral. Then $s_{n-1} = s_n^2 - x$ is integral, proving the claim.

So in particular s_1 and s_2 are integers and $s_2^2 = s_1^2 + s_1$. But if $s_1 > 0$, then $s_1^2 < s_1^2 + s_1 < (s_1 + 1)^2$, which is impossible. Similarly $s_1 < 0$ is impossible. So the only possible solution is $s_1 = 0$ and hence $x = 0$ and $y = 0$.

47.

Let the diagonals meet at O. Then CC'O is similar to AA'O (because CC'O = AA'O = 90, and COC', AOA' are opposite angles), so A'O/C'O = AO/CO. Similarly, B'O/D'O = BO/DO. AA'O is also similar to BB'O, so A'O/B'O = AO/BO. Thus OA':OB':OC':OD' = OA:OB:OC:OD. Hence triangles OA'B' and OAB are similar. Likewise OB'C' and OBC, OC'D' and OCD, and OD'A' and ODA. Hence result.

48.

Answer: n = 4 or prime.

If n = rs, with 1 < r < s, then r < s < n, and hence $rsn = n^2$ divides n!. Similarly, if $n = r^2$ with r > 2, then r < 2r < n, and hence n^2 divides n!. This covers all possibilities except n = 4 or n = prime, and it is easy to see that in these cases n^2 does not divide n!.

49.

```
    ./ \./ \.
    |   |   |
  ./°\./°\./°\.
  |   |   |   |
./°\./°\./°\./
|   |   |   |
*\./°\./°\./°
```

```
directions   1      2
                |      /
             *    *      *
                          \
                           3
```

Suppose vertex A is that marked * at the bottom left. Without loss of generality, B is in a 60 degree sector as shown. Assume the edges have unit length. The vertices can be partitioned into two sets (marked ° and . in the diagram). Each set forms a skewed lattice with axes at 60 degrees. Any path must alternate between the two lattices.

If B is on the same lattice as A, then we can give B coordinates (m,n) relative to A and the shortest path from A to B must move m units east and n units east of north. The shortest path between a lattice point and the next lattice point east is evidently one edge in direction 3 followed by one edge in direction 2. Similarly, the shortest path between a lattice point and the next lattice point east of north is one edge in direction 1, followed by one edge in direction 2. So a shortest path from A to B must have m+n edges in direction 2.

B is a distance $\sqrt{3}(m+n/2)$ east of A and a distance 3n/2 north of A, so $AB^2 = (3m^2+3mn+3n^2) < (4m^2+8mn+4n^2) = 4(m+n)^2$. So in this case the bug must travel more than AB/2 in direction 2.

Now suppose B is on the other lattice. Let C be the lattice point immediately north of A and D the lattice point in direction 3 from A. Then a shortest path from A to B must either be A to C and then a shortest path from C to B, or A to D and then a shortest path from D to B. Take B to have coordinates (m, n) relative to C or D.

In the first case, $AB^2 = (\sqrt{3}(m+n/2))^2 + (3n/2 + 1)^2 = (3m^2 + 3mn + 3n^2) + 3n + 1$ and a shortest path has m+n units in direction 2. But $4(m + n)^2 > (3m^2 + 3mn + 3n^2) + 3n + 1$, if $m^2 + n^2 + 5mn > 3n + 1$, which is true for m, n at least 1. If m=0 and n=1, then a shortest path has 2 units in direction 1 and $AB = \sqrt{7} < 4$. If m=1 and n=0, then AB=2 and a shortest path has 1 unit in each direction. So in this case (the only one so far) we have equality.

It remains to consider the case where the path starts out towards D. In this case $AB^2 = (\sqrt{3}(m+n/2) + \sqrt{3}/2)^2 + (3n/2 - 1/2)^2 = (3m^2+3mn+3n^2) + 3m + 1$ and a path has m + n units in direction 2. But $4(m + n)^2 > (3m^2 + 3mn + 3n^2) + 3m + 1$ for $m^2+n^2 + 5mn > 3m + 1$, which is true for m, n at least 1. If m=1, n=0, then a shortest path has 2 units in direction 3 and $AB = \sqrt{7} < 4$. Finally, if m=0 and n=1, then a shortest path has 1 unit in each direction and

AB = 2.
Thus the answer to the final question is 3, because the only cases where the bug travels exactly AB/2 in one direction are where it goes to the opposite vertex of a hexagon it is on.

50.
Let AB touch the circle at W, BC at X, CD at Y, and DA at Z. Then AO bisects angle ZOW and BO bisects angle XOW. So angle AOB is half angle ZOX. Similarly angle COD is half angle XOZ and hence angle AOB + angle COD equals 180.

51.
We have $k^n - a = b^n - a$ (mod $b - k$). Hence $b^n - a = 0$ (mod $b - k$) for every k not equal to b. But if b^n does not equal a, then by taking $k - b > b^n - a$ we could render the equation false.

52.
Answer 2^{n-2}. a_1 must be in the numerator, and a_2 must be in the denominator, but the other symbols can be in either. This is easily proved by induction.

53.
Answer: 5.
Tetrahedral faces are triangular, so each cube face requires at least two tetrahedral faces. So at least 12 tetrahedral faces are needed in all. At most three faces of a tetrahedron can be mutually orthogonal (and no two faces can be parallel), so at most 3 faces from each tetrahedron can contribute towards these 12. So we require at least 4 tetrahedra to provide the cube faces. But these tetrahedra each have volume at most 1/6 (1/3 x face area x 1, and face area is at most 1/2). So if we have only 4 tetrahedra in total then their total volume is less than the cube's volume. Contradiction. Hence we need at least 5 tetrahedra.
It can be done with 5: lop off 4 non-adjacent corners to leave a tetrahedron. More precisely, take the cube as ABCDA'B'C'D' with ABCD horizontal, A' directly under A, B' directly under B and so on. Then the five tetrahedra are AA'BD, CC'BC, DD'A'C', BB'A'C', BDA'C'.

54.
(a) This one must have slipped through: 121!
(b) Answer: 16,36,121,484. Suppose the number has more than 2 digits. Write it as $(10m + n)10^r + s$, where $1 \le m \le 9$, $0 \le n \le 9$, $0 \le s < 10^r$. Then we have $k(m.10^r + s) = (10m + n)10^r + s$, for some $k > 1$.
s does not contain the digits 0 or 5, so 5 does not divide s. Hence 5 divides $k-1$, and so k must be 6, 11, or 16 (if k was 21 or more, then the rhs would be negative). Since 25 does not divide $k-1$, we must have r=1 and s is a single digit.
We look at each possibility for k in turn. k = 6 gives no solutions. k = 11 gives about two dozen multiples of 11 from 121 to 891. By inspection the only squares are 121 and 484. k = 16 gives 192, which is not a square.
In addition, there is the possibility of 2 digit solutions, which I had overlooked. It is easiest to check each of the 2 digit squares, thus finding the additional solutions 16, 36.

55.
A necessary and sufficient condition for ABCD to have an inscribed circle is AB + CD = BC + AD. So we have AB + CD = 2AD, which we use repeatedly. Extend DC to X so that BX is parallel to EC. Then DX = AB + CD = 2AD and the triangles DEC, AEB, DBX are similar. Let h be the perpendicular distance from AB to CD. The similar triangles give us the heights of DEC and AEB in terms of h.
$1/r_1$ = perimeter ABE/(2 area ABE) = (AB + 2EB)/(AB.height) = (AB + 2.BD.AB/(AB+CD))/(AB.h.AB/(AB+CD)) = 2(AD + BD)/(AB.h). Similarly, $1/r_3$ = 2(AD + BD)/(CD.h).
The area of AED = area ABD – area ABE = 1/2 AB.h.CD/(2AD), so $1/r_2 = 1/r_4$ = perimeter ADE/(2 area ADE) = (AD + BD)/(h.AB.CD/2AD), and $1/r_2 + 1/r_4$ = 2(AD + BD)/h 2AD/(AB.CD) = 2(AB + BD)/h (AB + CD)/(AB.CD) = $1/r_1 + 1/r_3$.

56.

(a) Answer: $-[n/2]$.

Let $A = (x_1 + ... + x_n)^2$, $B = x_1^2 + ... + x_n^2$. Then we must minimize $A - B$. For n even, we separately minimize A and maximize B by taking half the x's to be +1 and half to be –1. For n odd we can take [n/2] x's to be +1, [n/2] to be –1, and one to be 0. That minimizes A and gives B one less than its maximum. That is the best we can do if we fix $A = 0$, since $A = 0$ requires an even number of x's to be non-zero and hence at least one to be zero. If we do not minimize A, then since its value must be an integer, its value will be at least 1. In that case, even if B is maximized we will not get a lower total.

(b) Answer: $-[n/2]$. For n even, the same argument works. For n odd we can clearly get $-[n/2]$, so it remains to prove that we cannot get a smaller sum. Suppose otherwise, so that x_i is a minimal sum with sum less than $-[n/2]$. Let $x_n = x$, then the sum is $x(x_1 + ... + x_{n-1})$ + sum of terms x_ix_j with $1 \le i, j < n$. But this is less than the sum for n–1, so $x(x_1 + ... + x_{n-1})$ must be negative, and since it is minimal we must have $|x| = 1$. But the same argument shows that all the terms have modulus 1. We now have a contradiction since we know that the minimum in this case is $-[n/2]$.

57.

The first player always wins.

Let the board be:

```
. F .
S . S
. F .
```

We call the squares marked F the F–squares, the squares marked S the S–squares, and the remaining squares the neutral squares. The first player wins if the sum of the two cards on the F–squares exceeds the sum of the two cards on the S–squares. We also call the first player F and the second player S.

Let the cards be $a_1>a_2> ... >a_9$. Let $t_1 = a_1 + a_9$, $t_2 = a_2 + a_8$, $t_3 = a_3 + a_7$, $t_4 = a_4 + a_6$.

If $t_1 > t_2$, or $t_1 = t_2 > t_3$, or $t_1 = t_2 = t_3 \ge t_4$ (*), then F's strategy is to get a total of t_1 or better on the F–squares and to force S to a lower score on the S–squares. If (*) does not hold, then F's strategy is to force S to t_1 or lower, and to get a higher score.

If (*) holds, then F starts by playing a_1 to an F–square. S must play to the remaining F–square, otherwise F will play a_3 or better to it on his next move and win. So S must play a_9 to the remaining F–square, giving F a total of t_1.

Now if $t_1 > t_2$, then F forces S to t_2 or worse by playing a_8 to an S–square.

If $t_1 = t_2 > t_3$, then F forces S to t_3 or worse by playing a_2 to a neutral square. If S plays to an S–square, then he cannot do better than $a_3 + a_8$, which loses. So he plays a_8 to a neutral square. But now F plays a_3 to an S–square, and S cannot do better than t_3.

If $t_1 = t_2 = t_3 > t_4$, then F forces S to t_4 or worse. He starts by playing a_2 to a neutral square. If does not prevent F playing a_8 to an S–square on his next move, then he cannot do better than $a_3 + a_8$, which loses. So he must play a_8 to a neutral square. Now F plays a_3 to a neutral square. If S does not prevent F playing a_7 to an S–square on the following move, then he cannot do better than $a_4 + a_7$ which loses, so he plays a_7 to a neutral square. F now plays a_4 to an S–square. S cannot now do better than t_4, which loses.

Finally, if $t_1 = t_2 = t_3 = t_4$, then F proceeds as in the last case except that at the end he plays a_4 to the last neutral square instead of to an S–square. S now gets $a_5 + a_6$ on the S–squares, which loses.

If (*) does not hold, then F starts by playing a_9 to an S–square. If S does not play to the other S–square, then F will play a_7 or a_8 there on his next move and S will lose. So S must play a_1 to the other square, and gets a total of t_1. F now plays to get t_2, t_3 or t_4 on the F–squares.

If $t_1 < t_2$, then F plays a_2 to an F–square and so gets at least t_2 and wins.

If $t_1 = t_2 < t_3$, then F plays a_8 to a neutral square. If S does not prevent F playing $a_2$to an F–square on his next move, then F will get at least $a_2 + a_7$ and win. So S must play a_2 to a neutral square. Now F plays a_3 to an F–square and so gets at least t_3 on the F–squares and wins.

Finally, if $t_1 = t_2 = t_3 < t_4$, then F plays as in the previous case, except that at the end he plays a_7 to a neutral square instead of a_3 to an F–square. S must prevent F playing a_3 to an F–square the following move, or F gets at least $a_3 + a_6$ and wins. So S plays a_3 to a neutral square. F now plays a_4 to an F square and so must get at least t_4, which wins.

58.
ZY bisects the angle AYB, so AD/BD = AY/BY. Similarly, XY bisects angle BYC, so CE/BE = CY/BY. But AY = CY. Hence AD/BD = CE/BE. Hence triangles BDE and BAC are similar and DE is parallel to AC.
Let BY intersect AC at W and AX at I. I is the in–center. AI bisects angle BAW, so WI/IB = AW/AB. Now consider the triangles AYW, BYA. Clearly angle AYW = angle BYA. Also angle WAY = angle CAY = angle ABY. Hence the triangles are similar and AW/AY = AB/BY. So AW/AB = AY/BY. Hence WI/IB = AY/BY = AD/BD. So triangles BDI and BAW are similar and DI is parallel to AW and hence to DE. So DE passes through I.

59.
The total is made up of numbers of the form abcabc, and pairs of numbers abcxyz, xyzabc. The former is abc.1001 and the sum of the pair is 1001(abc + xyz). So the total is divisible by 1001 and hence by 13.

60.
Let the lighthouse be at L. Take time t = 0 at the moment the boat starts its run, so that at t = 0 it is at S a distance d from L, and thereafter it is at a distance less than d. Take A and B a distance d from L so that ALBS is a semicircle with diameter AB and S the midpoint of the arc AB. During the period to $t = 2.5\ \pi.d/v$ the boat has traveled a distance less than d, so it cannot reach AB. But it is a distance less than d from L, so it must be inside the semicircle. But during this period the beam sweeps across from LA to LB and so it must illuminate the boat.

61.
Every time a person is on duty he is paired with two other people, so if the arrangement were possible the number of pairs involving a particular person would have to be even. But it is 99.

62.
Let BC be the side parallel to XY, h the length of the altitude from A, and r the radius of the in–circle. Then XY/BC = (h – 2r)/h. But r.p = h.BC. So $XY = (p - 2BC)BC/p = (p^2/8 - 2(BC - p/4)^2)/p$. So the maximum occurs when BC = p/4 and has value p/8.

63.
Taking i = j = k, we have that $x_{ii} = 0$. Now taking j=k, we have that $x_{ij} = -x_{ji}$. Define $a_i = x_{i1}$. Then we have $x_{i1} + x_{1j} + x_{ji} = 0$. Hence $x_{ij} = a_i - a_j$.

64.
Yes. Place a grid of 900 points in 30 equally spaced rows and columns, so that each point is a distance 15/31 from its nearest neighbours (or 15/31 from the edge). This blocks all rectangles except those slimmer than 1/2. Those slimmer than 1/2 must have length at least 2, so we can block them with a smaller set of rows and columns containing more finely spaced points.
Label the rows 1–30. In each of the 7 rows 3, 7, 11, 15, 19, 23, 27 place an additional 31 points, so that each of these rows has 61 equally spaced points at a spacing of 15/62. Similarly for the columns. So in total we are placing an additional 2.7.31 = 434 points. Any rectangle of length >2 must encounter one of these rows (or columns) and hence must have width less than 1/4. This blocks any rectangle except those with width < 1/4.
In each of the 3 rows 7, 15, 23 place an additional 62 points, so that each of these rows has 123 equally spaced points at a spacing of 15/124. Similarly for the columns. So in total we are placing an additional 2.3.62 = 372 points. Any rectangle of length >4 must encounter one of these rows (or columns) and hence must have width less than 1/8. This blocks any rectangle except those with width < 1/8 and hence length > 8.
In row 15 place an additional 124 points, so that it has a total of 247 equally spaced points at a spacing of 15/247. Similarly for column 15. This requires an additional 248 points. Any rectangle which can fit through these gaps has area at most 15 x 15/247 < 1. So we have blocked all rectangles with area 1 or more and used 900 + 434 + 372 + 248 = 1954 points.

Ilan Mayer, who seems to solve these problems effortlessly, came up with a neater arrangement of points. He used narrowly spaced points along widely spaced diagonals: (k/15, k/15) for k = 1,2,...,224; ((28*n+k)/15, k/15) for n = 1,2,...,7, k = 1,2,...,224–28*n; (k/15, (28*n+k)/15) for n = 1,2,...,7, k = 1,2,...,224–28*n. The diagonals are spaced 28/15 apart, so the biggest rectangle that can be fitted between two diagonals has sides 15/15 less epsilon and 15/15 less epsilon. For example, take the vertices as (14/15 + e, e), (29/15 – e, e), (14/15 + e, 15/15 – e), (29/15 – e, 15/15 – e). If one allows a rectangle to touch points (in other words if one took the rectangles to exclude their boundaries) then this does not work – many 15 x 1/15 rectangles will fit. But one can add an additional point on each of the 15 lines, keeping the points on each line evenly spaced. That blocks rectangles without boundary and still has only 1821 points.

65.

We can take all a_i to lie in the range (0,1) and all b_i to be 0 or 1. The largest positive value of the sum of (a_i – b_i) for any subset is achieved by taking the subset of those i for which $b_i = 0$. Similarly, the largest negative value is achieved by taking those i for which $b_i = 1$. So the worst subset will be one of those two.

If $a_i < a_j$, then we cannot have $b_i = 1$ and $b_j = 0$ if the set of b_is is to minimise the maximum sum, because swapping them would reduce the sum of a's with b = 0 and the sum of (1 – a)'s with b = 1. So if we order the a's so that $a_1 \le a_2 \le ... \le a_n$, then a best set of b's is $b_i = 0$ for i ≤ some k, and $b_i = 1$ for i > k. [If some of the a_i are equal, then we can find equally good sets of b's do not have this form, but we cannot get a lower maximum sum by departing from this form.]

Let $L_i = a_1 + a_2 + ... + a_i$, and $R_i = a_{i+1} + a_{i+2} + ... + a_n$. As we increase i the sums L_i increase and the sums R_i decrease, so for some k we must have $L_k < R_k$, $L_{k+1} \ge R_{k+1}$. Either k or k+1 must correspond to the optimum choice of b's to minimise the maximum sum.

Now assume that the a's form a maximal set, in other words they are chosen so that the minimum is as large as possible. We show first that in this case $L_{k+1} = R_k$. Suppose $L_{k+1} < R_k$. Then we could increase each of $a_{k+1}, a_{k+2}, ... , a_n$ by epsilon. This would leave L_k unaffected, but slightly increase L_{k+1} and slightly reduce R_k. For small epsilon this does not change the value of k, but increases the smaller of L_{k+1} and R_k, thus increasing the minimum and contradicting the maximality of the original a's. Similarly, if $L_{k+1} > R_k$, we could decrease each of $a_1, a_2, ... , a_{k+1}$ by epsilon, thus slightly increasing R_k and reducing L_{k+1}.

Suppose not all of $a_1, a_2, ... , a_{k+1}$ are equal. Take i so that $a_i < a_{i+1}$. Now increase each of $a_1, a_2, ... , a_i$ by epsilon and reduce each of $a_{i+1}, a_{i+2}, ... , a_{k+1}$ by epsilon', with epsilon and epsilon' sufficiently small that we do not upset the ordering or change the value of k, and with their relative sizes chosen so that L_{k+1} is increased. R_k is also increased, so we contradict the maximality of the a's. Hence all $a_1, a_2, ... , a_{k+1}$ are equal. Similarly, we show that all of $a_{k+1}, ... , a_n$ are equal. For if not we can increase slightly $a_{k+1}, ... , a_j$ and reduce slightly $a_{j+1}, ... , a_n$ to get a contradiction.

So we have established that all the a's must be equal. Suppose n is odd = 2m+1 and that all the a's equal x. Then for the optimum k we have (k+1)x = (2m+1–k)(1–x), hence k+1 = (2m+2)(1–x) and the maximum difference is (k+1)x = (2m+2)(1–x)x. This is maximised by taking x = 1/2, k = m, and is (m+1)/2 = (n+1)/4. If n is even = 2m, then for the optimum k we have (k+1)x = (2m–k)(1–x), so k+1 = (2m+1)(1–x), and the maximum difference is (k+1)x = (2m+1)(1–x)x. However, in this case we cannot take x = 1/2, because that would give k = m – 1/2 which is non-integral, so we take k = m–1 or m, both of which give a maximum difference of m(m+1)/(2m+1) = n(n+2)/(4n+4) < (n+1)/4.

66.

Disregard all edges except those used in the path from A to B, and for each of those let the multiplicity be the number of times it was traversed. Let the degree of a vertex be the sum of the multiplicities of its edges. The key is to notice that the degree of every vertex except A and B must be even. For as we traverse the path from A to B we increase the degree by 2 each time we pass through a vertex. But at the start of the path, as we leave A, we only increase its degree by 1. Similarly as we arrive at B for the last time.

Now construct a path from B as follows. Since B has odd degree it must have an edge of odd multiplicity. Suppose the edge connects B to C. Follow that edge and reduce its multiplicity by one, so that B's degree and C's degree are each reduced by one. Now C has odd degree, so it must have an edge of odd multiplicity. Repeat. Since there are only finitely many edges we must eventually be unable to continue the path. But the only way that can happen is if we reach A.

67.

(a) Each meeting involves 10.9/2 = 45 pairs. So after 40 meetings, there have been 1800 pairs. We are told that these are all distinct. But if there are N people on the committee, then there are only N(N–1)/2 pairs available. For N=60, this is only 1770.
(b) A subcommittee of 5 has 5.4/2 = 10 pairs. So 31 subcommittees have 310 pairs, and these are all distinct, since no two people are on more than one subcommittee. But a committee of 25 only has 25.24/2 = 300 pairs available.

68.
Notice that 0 is good and all negative numbers are bad. Take c = rs – r – s. First c, is bad. For suppose otherwise: c = mr + ns. Then mr + ns = (s–1)r – s. Hence (s–1–m)r = (n+1)s, so r divides n+1. Say n+1=kr, and then s–1–m=ks, so m = (1–k)s – 1. But n+1 is positive, so k≥1, and hence m is negative. Contradiction.
If k is good, then c–k must be bad (otherwise c would be good). Suppose k is bad. Since r and s are relatively prime we can find integers a and b with ar + bs = 1 and hence integers m and n with mr + ns = k. Adding a multiple of sr – rs to both sides if necessary, this gives a pair m, n with mr + ns = k and m non-negative. Now take the pair with the smallest possible non-negative m. Then m ≤ s–1 (for otherwise m' = m–s, n' = n+r would be a pair with smaller non-negative m). Also n ≤ –1, otherwise k would be good. Now c – k = (s – 1 – m)r + (–n – 1)s and the coefficients s – 1 – m and –n – 1 are both non-negative, so c – k is good.
So exactly (rs – r – s + 1)/2 integers are bad.

69.
Answer: 18π sec.
Let C be the position of the spy–plane at the moment the missile is fired. Let B be the point a quarter of the way around the circle from C (in the direction the spy–plane is moving). Then the missile moves along the semi–circle on diameter AB and hits the plane at B.
To see this take a point P on the quarter circle and let the line AX meet the semi–circle at Q. Let O be the center of the semicircle. The angle BOQ is twice the angle BAQ, so the arc BP is the same length as the arc BQ. Hence also the arc AQ is the same length as the arc CP.

70.
If A and B are at the greatest distance, then they must be vertices. For suppose A is not a vertex. Then there is a segment XY entirely contained in the polyhedron with A as an interior point. But now at least one of angles BAX, BAY must be at least 90. Suppose it is BAX. Then BX is longer than BA. Contradiction.
Take a plane through A perpendicular to the line AB. Then the polyhedron must lie entirely on one side of the plane, for if Z lay on the opposite side to B, then BZ would be longer than BA. Now move the plane slightly towards B keeping it perpendicular to AB. The intersection of the plane and the polyhedron must be a small polygon. The polygon must have at least 3 vertices, each of which must lie on an edge of the polyhedron starting at A. Select three of these edges.
As the plane is moved further towards B, the selected vertices may sometimes split into multiple vertices or they may sometimes coalesce. In the former case, just choose one of the daughter vertices. In the latter case, let O be the point of intersection of the plane and AB. Let O' be the point of intersection at the last coalescence (or A if there was none). Then we have three paths along edges, with no edges in common, each of which projects onto O'O and hence has length at least O'O. Now select one or more new vertices to replace any lost through coalescence and repeat.

71.
The spacecraft flies at a constant height, so that it can see a circular spot on the surface. It starts at the north pole and spirals down to the south pole, overlapping its previous track on each circuit. The alien cannot move fast enough to cross the track before the next circuit, so it is trapped inside a reducing area surrounding the south pole.
The value of 10 is not critical, so we do not have to optimise the details. Take the height above the surface to be half the radius. Then a diameter of the spot subtends an angle $2\cos^{-1}(1/1.5)$ at the center of the planet. $1/1.5 < 1/\sqrt{2}$, so the angle is more than 90 degrees. The critical case is evidently when the spacecraft is circling the equator. Using suitable units, we may take the radius of the planet to be 1 and the spaceship

speed to be 1. Then the diameter of the spot is $\pi/2$. We take the overlap to be 2/3, so that each revolution the track advances $\pi/6$. If the planet flew in a circle above the equator, the distance for a revolution would be 2π 1.5 = 3π. The helical distance must be less than $3\pi + \pi/6 = 19\pi/6$. So the alien can travel a distance $19\pi/60 < 2/3\ \pi/2$ and is thus trapped as claimed.

72.
The key is to notice that no loops of size greater than two are possible. For suppose we have $A_1, A_2, \dots, A_n$ with A_i watching A_{i+1} for $0 < i < n$, and A_n watching A_1. Then the distance $A_{i-1}A_i$ is greater than the distance A_iA_{i+1} for $1 < i < n$, and the distance A_1A_n is less than the distance A_1A_2. Hence the distance A_1A_n is less than the distance $A_{n-1}A_n$ and so A_{n-1} is closer to A_n than A_1. Contradiction.
Pick any soldier. Now pick the soldier he is watching, and so on. The total number of soldiers is finite so this process must terminate with some soldier watching his predecessor. If the process terminates after more than two soldiers have been picked, then the penultimate soldier is watched by more than one soldier. But in that case there must be another soldier who is unwatched, because the number of soldiers equals the number of soldiers watching.
If the process terminates after just two soldiers, then we have a pair of soldiers watching each other. Now repeat on the remaining soldiers. Either we find a soldier watched twice (in which case some other soldier must be unwatched) or all the soldiers pair off, except one, since the total number is odd. But that soldier must be unwatched.

73.
(a) Suppose the points lie in the order A, B, C, D. If P lies on AD, then the result is trivial, and we have equality if P lies outside the segment AD. So suppose P does not lie on AD.
Let M be the midpoint of AD. Take P' so that P, M, P' are collinear and PM = MP'. Then we wish to prove that PA + AP' > PB + BP'. Extend P'B to meet PA at Q. Then P'A + AQ > P'Q, so P'A + AP > P'Q + QP. But QP + BQ > PB, so QP + QP' > PB + PB'. Hence result.

(b) Let the foot of the perpendicular from B, C onto AD be X, Y respectively. Suppose that N, the midpoint of XY, is on the same side of M, the midpoint of AD, as D. Then take P to be a remote point on the line AD, the opposite side of A to D, so that A, D, M and N are all on the same side of the line PAD from P. Then PA + PD = 2PM < 2PN ≤ PB + PC. Contradiction. So we must have N coincide with M. But we still have PA + PD = 2PM = 2PN < PB + PC, unless both B and C are on the line AD. So we must have B and C on the line AD and AB = CD. It remains to show that B and C are between A and D. Take P = B. Then if C is not between A and D, we have PC > PD (or PA), contradiction.

74.
No. The smallest square greater than x^2 is $(x+1)^2$, so we must have $y > 2x$. Similarly $x > 2y$. Contradiction.

75.
Rearrange the children in the back row into order, and rearrange the front row in the same way, so that each child stays in front of the same child in the back row. Denote heights in the back row by a_i and heights in the front row by b_i. So we have $a_1 \le a_2 \le \dots \le a_n$, and $a_i > b_i$ for $i = 1, 2, \dots, n$.
Now if $i < j$, but $b_i > b_j$, then we may swap b_i and b_j and still have each child taller than the child in front of him. For $b_i < a_i \le a_j$, and $b_j < b_i < a_i$. By repeated swaps we can get the front row into height order. [For example, identify the shortest child and swap him to the first position, then the next shortest and so on.]

76.
Let ABCD have n lattice points along the side AB. Then it has kn lattice points along the side AD. Let X be the first lattice point along AB after leaving A. A shortest path from X to C must involve a total of kn + n – 1 moves between lattice points, n – 1 in the direction AB and kn in the direction BC. Hence the total number of such paths is $(kn + n - 1)!/((kn)!\ (n - 1)!)$. Similarly, the number of paths starting out along AD is $(kn + n - 1)!/((kn - 1)!\ n!)$. Let $m = (kn + n - 1)!/((kn - 1)!\ (n - 1)!)$. Then the number starting along AB is $m/(kn)$ and the number starting along AD is m/n, which is k times larger, as required.

77.

We show that you can pick $b_n, b_{n-1}, \dots, b_r$ so that $s_r = b_n a_n + b_{n-1} a_{n-1} + \dots + b_r a_r$ satisfies $0 \le s_r \le a_r$. Induction on r. Trivial for r = n. Suppose true for r. Then $-a_{r-1} \le s_r - a_{r-1} \le a_r - a_{r-1} \le a_{r-1}$. So with $b_{r-1} = -1$ we have $|s_{r-1}| \le a_{r-1}$. If necessary, we change the sign of all $b_n, b_{n-1}, \dots, b_{r-1}$ and obtain s_{r-1} as required. So the result is true for all $r \ge 1$ and hence for r = 1.

78.

Draw a rectangle width A/P on the inside of each side. The rectangles at each vertex must overlap since the angle at the vertex is less than 180. The total area of the rectangles is A, so the area covered must be less than A. Hence we can find a point not in any of the rectangles. But this point must be a distance more than A/P from each side, so we can use it as the center of the required circle.

79.

Take any path from A to B. Suppose it is $A=A_0, A_1, \dots, A_n=B$. We show by induction on r that we can find two disjoint paths from A to A_r. If r = 1, then take any vertex C distinct from A and A_1. Take any path from A_1 to C which does not go through A. Now take any path from C to A which does not go through A_1. Joining these two paths together gives a path p from A to A_1 which does not involve the edge AA_1. Then p and the edge AA_1 are the required disjoint paths.

Suppose now we have two disjoint paths $A, B_1, B_2, \dots, B_s, A_r$ and $A, B_t, B_{t-1}, \dots, B_{s+1}, A_r$ and we wish to find two disjoint paths joining A and A_{r+1}. Take a path between A and A_{r+1} which does not include A_r. If it also avoids all of $B_1, \dots, B_t$, then we are home, because it is disjoint from the alternative path $A, B_1, B_2, \dots, B_s, A_r, A_{r+1}$. If not, let B_i be the first of the B's on the path as we move from A_{r+1} to A. This allows us to construct two disjoint paths from A to A_{r+1}. One path goes from A to B_i and then from B_i to A_{r+1}. The other path goes around the other way to A_r and then along the edge to A_{r+1}. [Explicitly, if $i \le s$, then the paths are $A, B_1, B_2, \dots, B_i, \dots$ (new path) $\dots A_{r+1}$ and $A, B_t, B_{t-1}, \dots, A_r, A_{r+1}$. If $i > s$, then the paths are $A, B_t, B_{t-1}, \dots, B_i, \dots$ (new path) $\dots A_{r+1}$ and $A, B_1, \dots, B_s, A_r, A_{r+1}$.] Hence, by induction, there are two disjoint paths from A to B.

80.

Answer: the triangle DEF with FAE parallel to BC, DBF parallel to CA and DCE parallel to AB.

Let α be the angle between planes ABC and PBC. Let h be the perpendicular distance from H to the line BC, and let h_A be the perpendicular distance from A to the line BC. Then $PH = h \tan \alpha$, and the altitude from A to PBC is $h_A \sin \alpha$. Hence if PH is shorter than the altitude from A we require that $h < h_A \cos \alpha < h_A$. Similar arguments apply for B and C. So if PH is the shortest then H lies within triangle DEF.

If H does lie within DEF, then if we make α sufficiently small we will have $h < h_A \cos \alpha$ and hence PH will be shorter than the altitude from A. Similarly we can make PH sufficiently short that PH is less than the altitudes from B and C. Hence the inside of DEF is the required locus.

91.

a) True. Black moves to one end of a main diagonal and then moves along the diagonal to the opposite end. Each of the 499 rooks is in some row. Since black moves through each row, every rook must change row. But each of the rooks is also in some column and so every rook must also change column. A rook cannot change row and column in the same move, so white must make at least 998 moves before black reaches the opposite end of the diagonal. But it cannot start until black is two moves from its starting position, because if it moves a rook into row (or column) one or two earlier, then black is checked or can move into check. So it has only 997 moves available, which is one too few.

(b) False. Suppose the contrary, that after move n, the king is always in check after its move. Let the corners of the board be A, B, C, D. After move n, white moves all its rooks inside a square side 23 at corner A. The king must now be in the 23 rows between A and B or in the 23 columns between A and D. Suppose the latter. Then white moves all its rooks inside a square side 23 at corner B. This should take 499 moves. However, it could take longer if black used his king to obstruct the move. The worst case would be 3 x 23 additional moves (the king can only obstruct one row of 23 rooks, and each rook in the obstructed row could take 4 moves instead of one to reach its destination.). During this period the king must remain in the 23 rows from A to B or the 23 columns from A to D, since it must remain in check. Thus it cannot get to B

by the completion of the process. In fact, it must be at least 999 – 46 (the total number of moves required) – (499 + 69) (the number of moves available) = 385 moves behind.
White now moves all the rooks inside a square side 23 at corner C. The king cannot cut across (or it will be unchecked). It must keep within 23 squares of the edge. So it ends up 770 moves behind (more in fact, since it cannot obstruct the move as effectively). Finally, white moves all the rooks inside a square side 23 at corner D. The king cannot get to the side CD by the time this process is completed. So there is then a lag of over two hundred moves before it can get back into check. Note that it does not help black to change direction. Whatever black does, white ends up with all the rooks at a corner and the king a long way from the two checked sides.
(c) False. This follows from (b). But we may also use a simpler argument. Take coordinates x = 1 to 1000, y = 1 to 1000. White gets its pieces onto (2,0), (4,0), ... , (998,0). If the king moves onto (2n,*), then white moves its rook from (2n,0) to (2n–1,0), leaving the king unchecked. If the king moves to (2n–1,*) or (2n+1,*), then white moves its rook back to (2n,0), leaving the king unchecked. If the king stays on the line (2n,*), then white fills in time by toggling one of its endmost rooks to an adjacent square (and the king remains unchecked). The only way the black king can escape this repeated unchecking is by moving up to the line y = 0. If it does so, then white transfers all its rooks to the line y = 1000 and repeats the process. The transfer takes 499 moves. It takes black 1000 moves to follow, so during the 501 moves before black catches up, the king is subject to repeated unchecking.

92.
Answer: 2 1/3.
Let the square be ABCD. Let the vertices of the rhombus be P on AB, Q on AD, and R on BC. We require the locus of the fourth vertex S of the rhombus. Suppose P is a distance x from B. We may take $x \le 1/2$, since the locus for $x > 1/2$ is just the reflection of the locus for $x < 1/2$. Then since PR is parallel to QS, S is a distance x from the line AD. Also, by continuity, as Q varies over AD (with P fixed a distance x from B), the locus of S is a line segment.
The two extreme positions for S occur when Q coincides with A and when R coincides with C. When Q coincides with A the rhombus has side 1–x. Hence $BR^2 = (1-x)^2 - x^2 = 1 - 2x$. In this case SR is parallel to AB, so the distance of S from AB is $\sqrt{(1-2x)}$. When R coincides with C, the rhombus has side $\sqrt{(1+x^2)}$, so $AQ^2 = 1 + x^2 - (1-x)^2 = 2x$. Hence the distance of S from AB is $1 + \sqrt{(2x)}$.
Thus the locus of S over all possible rhombi is the interior of a curvilinear quadrilateral with vertices MDNC, where M is the midpoint of AB and N is the reflection of M in CD. Moreover the curve from M to C is just the translate of the curve from D to N, for if we put y = 1/2 – x, then $\sqrt{(1-2x)}$ becomes $\sqrt{(2y)}$. Thus if L is the midpoint of CD, then the area in the MLC plus the area in DLN is just 1/2, and the total area of the curvilinear quadrilateral is 1.
However, the arrangement of the vertices discussed above is not the only one. The order of vertices above is PQSR. We could also have PQRS or PSQR. In either case QR is a side rather than a diagonal of the rhombus. We consider the case PQRS (the case PSQR is just the reflection in the line MN). As before it is convenient to keep P fixed, but this time we take x to be the distance AP. Take y to be the distance AQ.
As before we find that S must lie on a line parallel to BC a distance x from it (on the other side to AD). Again we find that for fixed P, the locus of S is a segment of this line. If we assume that AQ > BR, then the two extreme positions are (1) QR parallel to AB, giving S on the line AB, (2) Q at D, giving S a distance x from the line AB. So as x varies from 0 to 1 we get a right–angled triangle sides 1, 1 and $\sqrt{2}$ and area 1/2.
However, we can also have BR > AQ. This gives points below the line AB. The extreme position is with R at C. Suppose QD = y. Then $1 + y^2 = x^2 + (1 - y)^2$, so $y = x^2/2$. This gives S a distance y below the line AB. This gives an additional area of 1/6 (by calculus – integrate $x^2/2$ from 0 to 1; I do not see how to do it without).
The triangle and the curvilinear triangle together form a curvilinear triangle area 1/2 + 1/6 = 2/3. There is an identical triangle formed by reflection in MN. Thus the total area is 1 + 2/3 + 2/3 = 2 1/3.

93.
Let r(m) denote the number obtained from m by reversing the digits.
We show first that k cannot be divisible by 2 or 5. It cannot be divisible by both, for then it ends in a zero and hence r(k) < k and so is not divisible by k (contradiction). So if 5 divides k, then the last digit of k must

be 5. Since r(k) is divisible by 5 its last digit must also be 5, so the first digit of k is 5. But now 3k has first digit 1 ($3.5 > 10$ and $3.6 < 20$), so r(3k) has last digit 1 and cannot be divisible by 5. Contradiction. If 2 divides k, then every multiple of k must be even. So the last digit of r(k) must be even and hence the first digit of k must be 2, 4, 6, or 8. If 2, then 5k has first digit 1, so r(2k) is odd. Contradiction. Similarly, if the first digit is 4, 3k has first digit 1; if 6, then 5k has first digit 3; if 8, then 2k has first digit 1. Contradiction. So k is not divisible by 2 or 5.

Suppose $k = 10^n a_n + ... + a_0$. k divides r(k), so $a_0 \geq 1$. Hence $(10^{n+1} - 1)k = 10^{2n+1}a_n + ... + 10^{n+1}a_0 - (10^n a_n + ... + a_0) = 10^{2n+1}a_n + ... + 10^{n+1}(a_0-1) + 10^n c_n + ... + 10c_1 + (c_0+1)$, where $c_i = 9 - a_i$. The reverse of this, $10^{2n+1}(c_0+1) + 10^{2n}c_1 + ... + 10^{n+1}c_n + 10^n(a_0-1) + ... + a_n$, is also divisible by k. So is the reverse of k, $10^n a_0 + ... + a_n$ and hence also their difference: $10^n(10^{n+1}(c_0+1) + 10^n c_1 + ... + 10c_n - 1)$. k has no factors 2 or 5, so k must divide $10^{n+1}(c_0+1) + 10^n c_1 + ... + 10c_n - 1$. Adding 10k, we find that k also divides $10^{n+2} + 10^n 9 + ... + 10.9 - 1 = 1099...989$ (n – 2 consecutive 9s) $= 11(10^{n+1} - 1)$.

We can now carry out exactly the same argument starting with $(10^{n+2} - 1)k$. This leads to k dividing $10^{n+2}(c_0+1) + ... + 10^2 c_0 + 10.9 - 1$ and hence also $10^{n+3} + 10^{n+1}9 + ... + 10^2 9 + 10.8 + 9 = 11(10^{n+2} - 1)$. Subtracting 10 times this from the previous number we conclude that k must divide $11(10^{n+1} - 1) - 11(10^{n+1} - 10) = 99$.

Finally, we note that any factor of 99 has the required property. For 3 and 9 divide a number if and only if they divide its digit sum. So if m is divisible by 3 or 9, then the number formed by any rearrangement of its digits is also divisble by 3 or 9. m is divisible by 11 if and only if the difference between the sums of alternate digits is divisible by 11, so if m is divisible by 11, then so is its reverse.

94.

Extend the sides to form two rectangles. Let the sides of the octagon have length a, b, c, d, e, f, g, h. Then we can find the rectangle sides. For example, one of the rectangles has opposite sides $a + (b + h)/\sqrt{2}$ and $e + (d + f)/\sqrt{2}$. Hence either $a = e$ or $\sqrt{2} = (b + h - d - f)/(a - e)$. The root is irrational, so we must have $a = e$. Similarly for the other pairs of opposite sides.

95.

$17^2 = 289 > 9.31$. So $17^{14} > 9^7\, 31^7$. But $3^7 = 2187 > 31^7$. Hence $17^{14} > 31^{11}$.

96.

Take compass directions aligned with the grid. Let N, E, S, W be the most northerly, easterly, southerly and westerly points on the circle. The arc from N to E must cross 100 north–south grid lines and 100 east–west grid lines. Each time it crosses a grid line it changes square (and it never crosses two grid lines at once, because it does not pass through any lattice points), so the arc N to E must pass through 200 in addition to the starting square. Similarly for the other 4 arcs. So the circle passes through a total of 800 squares (we count the starting square in the last 200).

97.

Let EF denote the number of students speaking English and French. Similarly define ES, FS, E, F, S, EFS. Then ES + EF + E + EFS = 50, EF + FS + F + EFS = 50. Subtracting: ES – F = FS – E. Similarly, ES – F = EF – S.

Pair off members of FS with members of E. Similarly, members of ES with F,and members of EF with S. The resulting pairs have one person speaking each language. If ES = F, then the only remaining students are those in EFS, who speak all three languages. We thus have a collection of units (pairs or individuals) each containing one speaker of each language.

If ES < F, then after the pairing off we are left with equal numbers of members of E, F, and S. These may be formed into triplets, with each triplet containing one speaker of each language. As before we also have the students in EFS. Again, we have partitioned the student body into units with each unit containing one speaker of each language.

If ES > F, then after the pairing off, we are left with an equal number of members of ES, FS and EF. These may be formed into triplets, with each triplet containing two speakers of each language. So, in this case we partition the student body into units with each unit containing either one speaker of each language, or two speakers of each language.

Finally, we may divide the units into 5 groups with 10 speakers of each language in each group.

98.
lhs = $1/(x-1) - 1/(x+1) + 1/(x-2) - 1/(x+2) + ... + 1/(x+10) - 1/(x-10) =$
$1/(x-1) - 1/(x+10) + 1/(x-2) - 1/(x+9) + ... + 1/(x-10) - 1/(x+1) =$ rhs.

99.
Answer: $n = 9$.
For $n < 6$, there is at most one length of diagonal. For $n = 6, 7$ the longest and shortest, and a side of the n–gon form a triangle, so the difference between the longest and shortest is less than the side.
For $n > 7$ the side has length $2R \sin \pi/n$, the shortest diagonal has length $2R \sin 2\pi/n$, and the longest diagonal has length $2R$ for n even and $2R \cos \pi/2n$ for n odd (where R is the radius of the circumcircle). Thus we require:
$\sin 2\pi/n + \sin \pi/n = 1$ and n even, or
$\sin 2\pi/n + \sin \pi/n = \cos \pi/2n$ and n odd.
Evidently the lhs is a strictly decreasing function of n and the rhs is an increasing function of n, so there can be at most one solution of each equation. The second equation is satisfied by $n = 9$, although it is easier to see that there is a quadrilateral with the longest diagonal and shortest diagonals as one pair of opposite sides, and 9–gon sides as the other pair of opposite sides. The angle between the longest side and an adjacent side is 60, so that its length is the length of the shortest diagonal plus 2 x 9–gon side x cos 60. Hence that is the only solution for n odd.
For $n = 8$ we have the same quadrilateral as for the 9–gon except that the angle is 67.5 and hence the difference is less than 1. For $n = 10$, $\sin 2\pi/10 + \sin \pi/10 = \sin \pi/10$
$(2\cos \pi/10 + 1) < 3 \sin \pi/10 < 3 \pi/10 < 1$. So there are no solutions for n even ≥ 10, and hence no solutions for n even.